Preface

At the turn of the twentieth century mathematicians had a monopoly on well defined complicated global problems, e.g. celestial mechanics, fixed points of high dimensional nonlinear functions, the geometry of level sets of differentiable functions, algebraic varieties, distinguishing topological spaces, etc. Algebraic topology, of which homology is a fundamental part, was developed in response to such challenges and represents one of the great achievements of twentieth century mathematics. While its roots can be traced to the middle of the nineteenth century with Euler's famous formula that for the surface of a convex polyhedron

$$\text{faces} - \text{edges} + \text{vertices} = 2,$$

it is fair to say that it was began as a subject in its own right in the seminal works of Henri Poincaré on "Analysis Situs." Though motivated by analytic problems as his techniques developed they took on a combinatorial form similar in spirit to Euler's formula.

The power of algebraic topology lies in it coarseness. To understand this statement consider Euler's formula. Observe for instance that the size of the polyhedron is of no importance. In particular, therefore, small changes in the shape of the polyhedron do not alter the formula. On the other hand if one begins with a polyhedron that is punctured by k holes as opposed to a convex polyhedron, then the formula becomes

$$\text{faces} - \text{edges} + \text{vertices} = 2 - 2k.$$

As a result, counting local objects, the faces, edges and vertices, of a polyhedron allows us to determine a global property, how many holes it has. Furthermore, it shows that a formulas of this form can be used to distinguish objects with important different geometric properties.

The potential of Poincaré's revolutionary ideas for dealing with global problems were quickly recognized and this led to a broad development of the subject. However, as is to be expected, the form of development matched the problems of interest. As indicated above a typical question might be concerned with the structure of the level set of a differentiable function. Solving such problems using purely combinatorial arguments suggested by Euler's formula, i.e. cutting the set into a multitude of small pieces and then counting, is in general impractical. This led to the very formidable and powerful algebraic machinery that is now referred to as Algebraic Topology. In its simplest form this tool takes objects defined in terms of traditional mathematical formulae and produces algebraic invariants which provide fundamental information about geometric properties of the objects.

As we begin the twenty-first century complexity has spread beyond the realm of mathematics. With the advent of computers and sophisticated sensing devices, scientists, engineers, doctors, social scientists, businessmen, all have access to or through numerical simulation can create huge data files. Furthermore, for some of these data sets the crucial information is geometric in nature, but it makes little sense to think of these geometric objects as being presented in terms of or derived from traditional mathematical formulae. As an example think of medical imaging. Notice that even though the input is different, the problems remain the same identifying and classifying geometric properties or abnormalities. Furthermore, inherent in numerical or experimental data is error. Thus what is needed is a framework in which geometrical objects can be recognized even in the presence of small perturbations.

Hopefully these arguments suggest that the extraordinary success of algebraic topology in the traditional domains of mathematics can be carried over to this new set of challenges. However, to do so requires the ability to efficiently compute the algebraic topological quantities starting with experimental or numerical data - information which is purely combinatorial in nature.

The purpose of this book is to present a computational approach to homology with the hope that such a theory will prove beneficial to the analysis and understanding of today's complex geometric challenges. Naturally this means that our intended audience includes computer scientists, engineers, experimentalists and theoreticians in nonlinear science. As such we have tried to keep the mathematical prerequisites to an absolute minimum. At the same time we, the authors, are mathematicians and proud of our trade. We believe that the most significant applications of the theory will be realized by those who understand the fundamental concepts of the theory. Therefore, we have insisted on a rigorous development of the subject. Thus, this book can also be used as an introductory text in homology for mathematics students. It differs from the traditional introductory topology books in that a great deal of effort is spent on discussing the computational aspects of the subject.

The book is divided into three parts. Part I contains the core material. It provides a rigorous introduction to the homology of spaces and continuous functions and is meant to be read in sequence. Homology is a beautiful subject in that topology is transformed into algebra and from the algebra one can recover aspects of the topology. However, this process involves some deep ideas and it is easy to lose track of the big picture in the midst of the mathematical technicalities. With this in mind we have included two preview chapters (Chapters 1 and 5) where the ideas are sketched through very simple examples and without too much concern for rigor.

In Chapter 2 we define homology and investigate its most elementary properties. In particular, we explain how to each topological space we can assign a sequence of abelian groups call the homology groups of the space. There is a caveat and that is that the topological spaces we consider must be built out of d-dimensional unit cubes with vertices on the integer lattice. This is in contrast to the standard combinatorial approach which is based on simplices. There are two reasons for this. The first comes from applications. Consider a digital images. The basic building blocks are pixels which are easily identified with squares. Similarly an experimental or numerically generated data point comes with errors. Thus an d-dimensional data point can be thought of as lying inside a d-dimensional cube whose width is determined by the error bounds. The second reason - which will be made clear in the text - comes from the simplicity of certain algorithms.

In Chapter 3 we show that homology is computable by presenting in detail an algorithm based on linear algebra over the integers. This is essential in the sense that it demonstrates that the homology group of a topological space made up of cubes is computable. However, for spaces made up of many cubes the algorithm is of little immediate practical value. Therefore, in Chapter 4 we introduce combinatorial techniques for reducing the number of elements involved in the computation.

The contents of Chapter 4 also naturally foreshadow questions concerning maps between topological spaces and maps between the associated homology groups. The construction of homology maps is done in Chapter 6. Again, we approach this problem from the point of view of multivalued maps. This has an extremely important consequence. One can efficiently compute homology maps - an absolutely essential feature given the purposes of this endeavor. We know of no practical algorithms for producing simplicial maps which approximate continuous maps. Algorithms for the computation of these maps are discussed in Chapter 7.

As was mentioned earlier we have attempted to keep the necessary prerequisites for this book to a minimum. Ideally the reader would have had an introductory course in point-set topology, an abstract algebra course and familiarity with computer algorithms. On the other hand such a reader would know far more topology, algebra and computer science than is really necessary. Furthermore, the union of these subjects in the typical curriculum of our desired readers is probably fairly rare. For this reason we have included in Part III brief introductions to these subjects. Perhaps a more traditional organization would have placed Chapters 12, 13, and 14 at the beginning of the book. However, it is our opinion that nothing kills the excitement of studying a new subject as rapidly as the idea of multiple chapters of preliminary material. We suggest that the reader begin with Chapter 1 and consult the last three Chapters on a need to know basis.

We argued at the beginning of this preface that algebraic topology has an important role to play in the analysis of numerical and experimental data. In Part II we elaborate on these ideas. It should be mentioned that applications of homology to these types of problems is a fairly new idea and still in a fairly primitive state. Thus the focus of this part is on conveying the potential rather than elaborate applications. We begin in Chapter 8 with a discussion that relates cubical complexes to image data and numerically generated data. Even the simple examples presented here suggest the need for more sophisticated ideas from homology. Therefore, Chapter 9 introduces more sophisticated algebraic concepts and computational techniques that are at the heart of homological algebra. In Chapter 10 we indicate how homology can be used to study nonlinear dynamics. In some sense this is one of the most traditional applications of the subject - the focus of much of Poincaré's work was related to differential equations and, in fact, he set the foundations for what is known today as dynamical systems. The modern twist is that these ideas can be tied to numerical methods to obtain computer assisted proofs in this subject. Finally, thought the cubical theory presented in this book has concrete advantages it is also in many ways too rigid. Thus in Chapter 11 we show that the cubical theory is equivalent to the simplicial theory, thereby tying the results and techniques of this book to the enormous body of work known as algebraic topology.

Preliminary version of this book have already been used by the authors to teach courses at a variety of levels: undergraduate and graduate; for a mixed audience of mathematics, computer science and engineering students; as a one-semester course or a full year course. As we mentioned earlier, Part I is meant as a core material. However, in a one-semester course, a teacher attempting to follow closely the order of chapters could not arrive at applications or extensions presented in Part II. It is possible to jump directly to specific topics in Part II without having completed Part I and the diagram in Figure 0.1 is indicating how. The arrows do not necessarily mean that the whole contents of a related chapter can be taught after skipping preceding chapters. For example, a teacher orienting his course towards applications may go immediately from Chapter 2 to Chapter 8 or from Chapter 6 to Chapter 10 provided that certain statements about properties and computation of homology are temporarily taken for granted. Also, a teacher may cover the first two sections of Chapter 11 concerning simplicial complexes right after Chapter 2 or even parallel to it but the last section on the homology functor could only be attained after having taught Chapter 6. Part III can be regarded either as an appendix or as a core material depending on the level of the audience and the profile of a course.

Many colleagues and students helped us by means of valuable comments, detailed proofreading or by research contributions which affluence the contents of the book. First, the research of our graduate students: shaped and

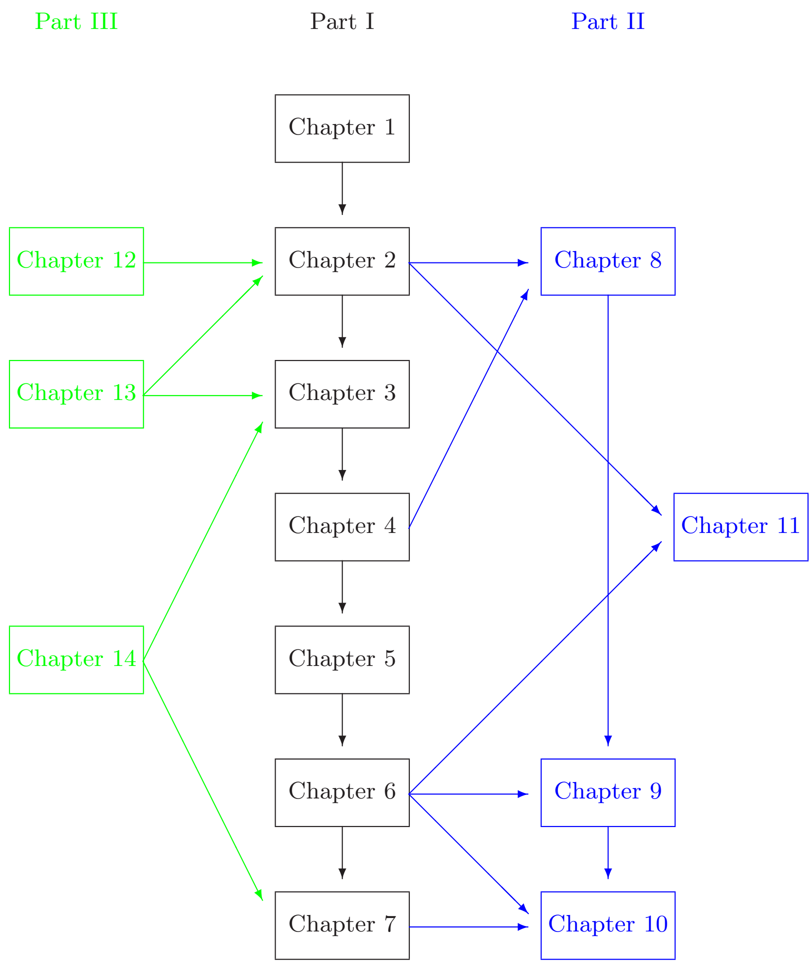

Fig. 0.1. Chapter Dependence Chart

motivated the whole project of writing this book. Teaching this material was exciting and challenging due to the freshness of results presented and the interplay between the mathematical rigor, computations, and applications. Thus the feedback we got from students enrolled in our courses played a very important role in improving the text. We are particularly indebted to: for a detailed proofreading of the text at its early stages.

Atlanta, Kraków, Sherbrooke,
January 2003

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Contents

Part I Homology

1

Preview of Homology

Homology is a very powerful tool in that it allows one to draw conclusions about *global* properties of spaces and maps from *local* computations. It also involves a wonderful mixture of algebra, combinatorics, computation, and topology. Each of these subjects is, of course, interesting in its own right, and appears as the subject of multiple sections in this book. But, our primary objective is to see how they can be combined to produce homology and it is easy to loose sight of this objective along the way. For this reason we begin with a preview that is meant to motivate. This chapter is intended to be read as if it were an entertaining puzzle or an intriguing mystery. Don't sweat the details - look for patterns and try to get a feeling for the big picture. Much of the rest of the book is devoted to filling in the details.

Let us begin by looking at some extremely simple pictures. In Figure 1.1(a) we see a line segment in the plane. If we think of this as a piece of fencing, it is clear that it does not enclose any region in the plane. On the other hand, Figure 1.1(b) clearly encloses the square $(0,1) \times (0,1)$. If we add more segments we can enclose more squares as in Figure 1.1(c), though, of course, some segments need not enclose any region. Finally, as is indicated by the shaded square in Figure 1.1(d), by filling in some regions we can eliminate enclosed areas.

The previous comments are so obvious as to border on being boring. So let us ask a more interesting question.

> *Can we develop an algebraic tool that tells us how many regions are enclosed by a set of line segments?*

For example, it would be nice to be able to enter Figure 1.2 into a computer and have the computer tell us how many bounded regions there are.

Of course, there is no reason to limit our questions to objects that lie in the plane. Figure 1.3 is a tomographic image of a section of a human heart. For a given point in time the entire tomographic data consists of 10 such images. Let us assume that for each such figure we can extract those pixels that correspond to the heart. Can we develop algorithms that would determine

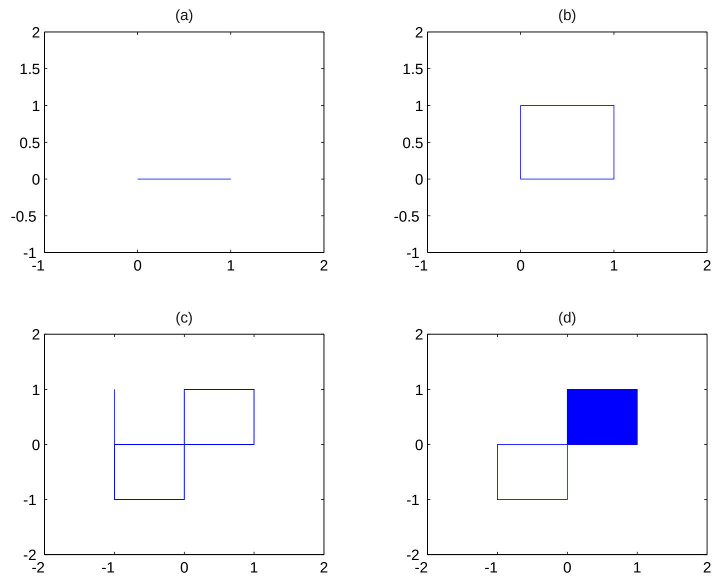


Fig. 1.1. (a) A simple line segment in the plane does not enclose any region. (b) These four line segments enclose the region $(0,1) \times (0,1)$. (c) It is easy to bound more than one region. (d) By filling in a region we can eliminate bounded regions.

how many holes, e.g. values or defects, the heart possesses? In fact, there are twenty such sets of figures for each heart beat. Could these algorithms be extended so that we could detect the opening and closing of the holes?

This book is all about developing such an algebraic tool. In particular, we will learn how to associate to each picture (which we will start calling a topological space) a sequence of algebraic objects called homology groups. These homology groups, in turn will provide us with geometric information about picture. Of course we have not told you what a homology group is - that is the point of this book. For the moment just remember the words.

1.1 Graphs

As was mentioned earlier, it would be nice to be able to enter complicated pictures into the computer and have it tell us how many bounded regions there are. However, we are still trying to motivate the subject, so let's keep things as simple as possible. In particular, let's stick to objects such as those of Figures 1.1 and 1.2.

To do mathematics we need to make sure that these simple objects are well defined. Graphs provide a nice starting point.

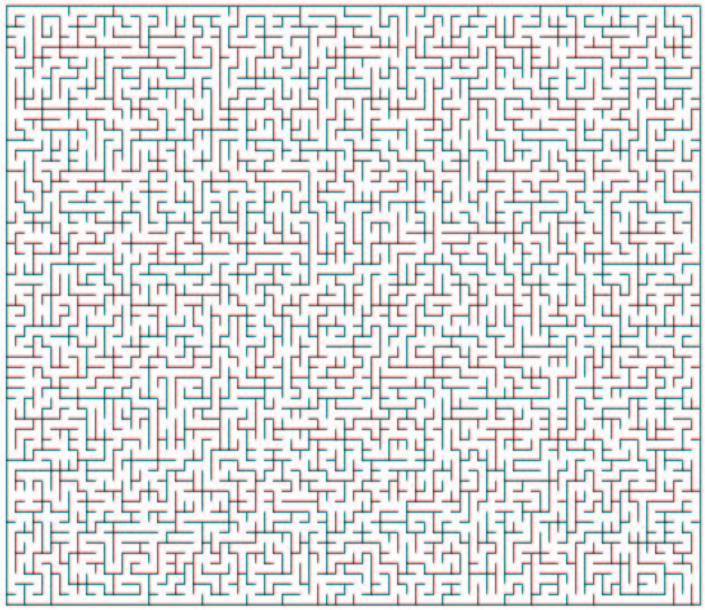


Fig. 1.2. How many distinct bounded regions are in this complicated maze?

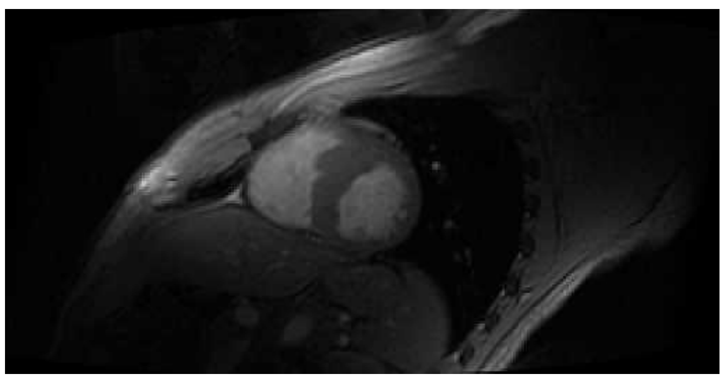


Fig. 1.3. A tomographic section of a human heart.

Definition 1.1 A *graph* G is a subset of $\mathbf{R}^3$ made up of a finite collection of points $\{v_1, \ldots v_n\}$, called *vertices*, together with straight line segments $\{e_1, \ldots, e_m\}$, joining vertices, called *edges* which satisfy the following intersection conditions:

1. the intersection of distinct edges is either empty or consists of exactly one vertex, and

2. if an edge and a vertex intersect, then the vertex is an endpoint of the edge.

More explicitly, an edge joining vertices v_0 and v_1 is the set of points

$$\{x \in \mathbf{R}^3 \mid x = tv_0 + (1-t)v_1,\ 0 \leq t \leq 1\}$$

which is denoted by $[v_0, v_1]$.

A *path* in G is an ordered sequence of edges of the form

$$\{[v_0, v_1], [v_1, v_2], \ldots, [v_{l-1}, v_l]\}.$$

This path *begins* at v_0 and *ends* at v_l. Its *length* is l, the number of its edges. A graph G is *connected* if for every pair of vertices $u, w \in G$, there is a path in G which begins at u and ends at w. A *loop* is a path which consists of distinct edges and begins and ends at the same vertex. A connected graph which contains no loops is a *tree*.

Observe that Figures 1.1(a), (b), (c) and Figure 1.2 are graphs. Admittedly they are drawn as subsets of $\mathbf{R}^2$, but we can think of $\mathbf{R}^2 \subset \mathbf{R}^3$. Figure 1.1(d) is not a graph. It is also easy to see that Figures 1.1(b) and (c) contain loops, and that Figures 1.1(a) is a tree. What about Figure 1.2?

Graphs, and more generally topological spaces, *cannot* be entered into a computer. As defined above a graph is a subset of $\mathbf{R}^3$ and so consists of infinitely many points, but a computer can only store a finite amount of data. Of course, there is a very natural way to *represent* a graph involving only of a finite amount of information.

Definition 1.2 A *combinatorial graph* is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set whose elements are called *vertices* and $\mathcal{E}$ is a collection of pairs of distinct elements of $\mathcal{V}$ called *edges*. If an edge e of a combinatorial graph consists of the pair of vertices v_0 and v_1, we will write $e = [v_0, v_1]$.

It may seem at this point that we are making a big deal out of a minor issue. Consider, however, the sphere, $\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. It is not so clear how to represent this in terms of a finite amount of data, but as we shall eventually see we can compute its homology via a combinatorial representation. In fact, homology is a combinatorial object, this is what makes it such a powerful tool.

Even in the setting of graphs, the issue of combinatorial representations is far from clear. Consider for example the graph $G = [0, 1] \subset \mathbf{R}$. How should we represent it as a combinatorial graph? Since we have not said what the vertices and edges are, the most obvious answer is to let $\mathcal{V} = \{0, 1\}$ and $\mathcal{E} = \{[0, 1]\}$. However, G could also be thought of as the graph containing the vertices 0, 1/2, 1 and the edges $[0, 1/2]$, $[1/2, 1]$, in which case the natural combinatorial representation is given by $\mathcal{V}_2 = \{0, 1/2, 1\}$ and $\mathcal{E}_2 = \{[0, 1/2], [1/2, 1]\}$. More generally, we can think of G as being made up of many short segments leading to the combinatorial representation

$$\mathcal{V}_n := \{j/n \mid j = 0, \dots n\}, \quad \mathcal{E}_n := \{[j/n, (j+1)/n] \mid j = 0, \dots, n-1\}.$$

We are motivating homology in terms of pictures, e.g. graphs. However, our input data to the computer will be a combinatorial graph. Thus, to prove that homology is an invariant of a picture we will have to show that given any two combinatorial graphs that represent the same set, the corresponding homology is the same. This is not trivial! In fact, it will not be proven until Chapter 11.

On the other hand, in this chapter we are not supposed to be worrying about details, so we won't. However, we should not forget that there is this potential problem:

> *Can we make sure that two different combinatorial objects which give rise to the same set also give rise to the same homology?*

Before turning to the algebra we want to prove a simple property about trees.

A vertex which only intersects a single edge is called a *free vertex*.

Proposition 1.3 *Every tree contains at least one free vertex.*

Proof. Assume not. Then there exists a tree T with 0 free vertices. Let n be the number of edges in T. Let e_1 be an edge in T. Label its vertices by v_1^- and v_1^+. Since T has no free vertices, there is an edge e_2 with vertices $v_2^\pm$ such that $v_1^+ = v_2^-$. Continuing in this manner we can label the edges by e_i and the vertices by $v_i^\pm$, where $v_i^- = v_{i-1}^+$. Since there are only a finite number of vertices, at some point in this procedure we get $v_i^+ = v_j^-$ for some $i > j \geq 1$. Then $\{e_j, e_{j+1}, \dots, e_i\}$ forms a loop. This is a contradiction. □

Exercises

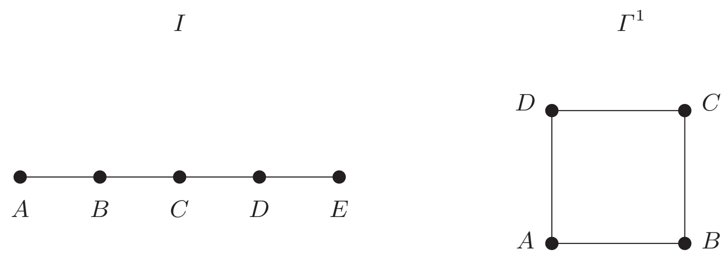
1.1 Associate combinatorial graphs to Figures 1.1(a), (b), and (c).

1.2 Among all paths joining two vertices v and w there is at least one which has minimal length. Such a path is called minimal. Show that any two edges and vertices on a minimal path are different.

1.3 Let T be a tree with n edges. Prove that T has $n + 1$ vertices.
Hint: Argue by induction on the number of edges.

1.2 Topological and Algebraic Boundaries

As was stated earlier our goal is to develop an algebraic means of detecting whether a set bounds a region or not. So we begin with the two simple sets of Figure 1.4, an interval I and the perimeter Γ^1 of a square. We want to think of these sets as graphs. As is indicated in Figure 1.4, we represent both sets by graphs consisting of four edges. The difference lies in the set of vertices.



$\mathcal{E}(I) = \{[A, B], [B, B], [C, D], [D, E]\}$

$\mathcal{V}(I) = \{A, B, C, D, E\}$

$\mathcal{E}(\Gamma^1) = \{[A, B], [B, C], [C, D], [D, A]\}$

$\mathcal{V}(\Gamma^1) = \{A, B, C, D\}$

Fig. 1.4. Graphs and corresponding combinatorial graphs for $[0, 1]$ and Γ^1

We mentioned earlier that homology has the remarkable property that local calculations lead to knowledge about global properties. As was already observed, the difference between the combinatorial graphs of I and Γ^1 is found in the vertices, which are clearly local objects. So let us focus on vertices and observe that they represent the end points or, as we shall call them from now on, the boundary points of edges.

Consider both the graph and the combinatorial graph of I. The left hand column of Table 1.1 indicates the boundary points of each of the edges. The right hand column is derived from the combinatorial graph. Recall that our goal is to produce an algebraic tool for understanding graphs. Instead of starting with formal definitions we are going to look for patterns. So for the moment the elements of the right hand column can be considered to be algebraic quantities that correspond to elements of the combinatorial graph.

Topology		Algebra
$\mathrm{bd}\,[A, B] = \{A\} \cup \{B\}$		$\partial\widehat{[A, B]} = \widehat{A} + \widehat{B}$
$\mathrm{bd}\,[B, C] = \{B\} \cup \{C\}$		$\partial\widehat{[B, C]} = \widehat{B} + \widehat{C}$
$\mathrm{bd}\,[C, D] = \{C\} \cup \{D\}$		$\partial\widehat{[C, D]} = \widehat{C} + \widehat{D}$
$\mathrm{bd}\,[D, E] = \{D\} \cup \{E\}$		$\partial\widehat{[D, E]} = \widehat{D} + \widehat{E}$

Table 1.1. Topological and algebraic boundaries in $[0, 1]$.

On the topological level such basic algebraic notions as addition and subtraction of edges and points is not an obvious concept. The point of moving to an algebraic level is to allow ourselves this luxury. To distinguish between sets and algebra, we write the algebraic objects with a hat on top and allow ourselves to formally add them. For example the topological objects such as a point $\{A\}$ and an edge $[A, B]$ become an algebraic object $\widehat{A}$ and $\widehat{[A, B]}$. Furthermore, we allow ourselves the luxury of writing down expressions like $\widehat{A} + \widehat{B}$, $\widehat{[A, B]} + \widehat{[B, C]}$ or even $\widehat{A} + \widehat{A} = 2\widehat{A}$.

How should we interpret the symbol ∂, called the *boundary operator*, which we have written in the table? We are doing algebra, so it should be some type of map that takes the algebraic object $\widehat{[A, B]}$ to the sum of $\widehat{A}$ and $\widehat{B}$. The nicest maps are *linear maps*. This would mean that

$$\begin{aligned}\partial(\widehat{[A, B]} + \widehat{[B, C]}) &= \partial(\widehat{[A, B]}) + \partial(\widehat{[B, C]}) \\ &\stackrel{(1)}{=} \widehat{A} + \widehat{B} + \widehat{B} + \widehat{C} \\ &= \widehat{A} + 2\widehat{B} + \widehat{C}\end{aligned}$$

where $\stackrel{(1)}{=}$ follows from Table 1.1.

On the other hand, we already pointed out that sets need not have unique representations as graphs. We could equally well have chosen a different combinatorial graph to represent I, for example

$$\mathcal{E}'(I) = \{[A, C], [C, D], [D, E]\} \quad \mathcal{V}'(I) = \{A, C, D, E\}.$$

In this case $\operatorname{bd}[A, C] = \{A\} \cup \{C\}$ and so $\partial\widehat{[A, C]} = \widehat{A} + \widehat{C}$. If we think that $+$ on the algebraic side somehow matches $\cup$ on the topology side, then this suggests that we would like

$$\partial(\widehat{[A, B]} + \widehat{[B, C]}) = \widehat{A} + \widehat{C} = \partial\widehat{[A, C]}.$$

The only way that this can happen is for $2\widehat{B} = 0$. This may seem like a pretty strange relation and suggests that at this point there are three things we can do:

(i) give up;
(ii) start over and try to find a different definition for ∂; or
(iii) be stubborn and press on.

The fact that this book has been written suggests that we are not about to give up. We shall discuss option (ii) in Section 1.3. For now we shall just press on and adopt the trick of counting *modulo 2*. This means that we will just check whether an element appears an odd or even number of times; if it is odd we keep the element, if it is even we discard the element, i.e. we declare

$$0 = 2\widehat{A} = 2\widehat{B} = 2\widehat{C} = 2\widehat{D} = 2\widehat{E}.$$

Hats, boundary operators, linear maps, counting modulo 2, have we justified any of this? No! For the moment we just want to see if we can observe any nice patterns - we are experimenting! Of course, if we wish to do mathematics we will eventually have to justify these ideas, but that will be done in the later chapters.

Continuing to use the presumed linearity of ∂ and counting modulo 2, we have that

$$\begin{aligned}\partial\left(\widehat{[A,B]}+\widehat{[B,C]}+\widehat{[C,D]}+\widehat{[D,E]}\right) &= \widehat{A}+\widehat{B}+\widehat{B}+\widehat{C}+\widehat{C}+\widehat{D}+\widehat{D}+\widehat{E}\\ &= \widehat{A}+\widehat{E}.\end{aligned}$$

As an indication that we are not too far off track observe that if we had begun with a representation of I in terms of the combinatorial graph

$$\mathcal{E}''(I)=\{[A,E]\}\quad \mathcal{V}''(I)=\{A,E\}$$

then $\operatorname{bd}[A,E]=\{A\}\cup\{E\}$.

Doing the same for the graph and combinatorial graph representing Γ^1 we get Table 1.2. Adding up the algebraic boundaries we have

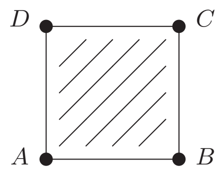
$$\begin{aligned}\partial\left(\widehat{[A,B]}+\widehat{[B,C]}+\widehat{[C,D]}+\widehat{[D,A]}\right) &= \widehat{A}+\widehat{B}+\widehat{B}+\widehat{C}+\widehat{C}+\widehat{D}+\widehat{D}+\widehat{A}\\ &= \widehat{A}+\widehat{A} \qquad (1.1)\\ &= 0.\end{aligned}$$

Topology		Algebra
$\operatorname{bd}[A,B]=\{A\}\cup\{B\}$		$\partial\widehat{[A,B]}=\widehat{A}+\widehat{B}$
$\operatorname{bd}[B,C]=\{B\}\cup\{C\}$		$\partial\widehat{[B,C]}=\widehat{B}+\widehat{C}$
$\operatorname{bd}[C,D]=\{C\}\cup\{D\}$		$\partial\widehat{[C,D]}=\widehat{C}+\widehat{D}$
$\operatorname{bd}[D,A]=\{D\}\cup\{A\}$		$\partial\widehat{[D,A]}=\widehat{D}+\widehat{A}$

Table 1.2. Topology and algebra of boundaries in Γ^1.

Based on these two examples one might make the extravagant claim that spaces with *cycles*, i.e. algebraic objects whose boundaries add up to zero, enclose regions. This is almost true.

To see how this fails, observe that we could "fill in" Γ^1 (think of Figure 1.1(d)). How does this affect the algebra? To make sense of this we need to go beyond graphs into *cubical complexes* which will be defined later. For the moment consider the picture and collection of sets in Figure 1.5. The new



$$\{Q\}$$

$$\mathcal{E}' = \{[A,B],[B,C],[C,D],[D,A]\}$$

$$\mathcal{V}' = \{A,B,C,D\}$$

Fig. 1.5. The set Q and corresponding combinatorial data.

aspect is the square Q which has filled in the region bounded by Γ^1. This is coded in the combinatorial information as the element $\{Q\}$.

Observe that the edge or boundary of Q is Γ^1. Table 1.3 contains the topological boundary information and the associated algebra.

Topology	Algebra
$\operatorname{bd} Q = [A,B] \cup [B,C] \cup [C,D] \cup [D,A]$	$\partial\widehat{Q} = \widehat{[A,B]} + \widehat{[B,C]} + \widehat{[C,D]} + \widehat{[D,A]}$
$\operatorname{bd}[A,B] = \{A\} \cup \{B\}$	$\partial\widehat{[A,B]} = \widehat{A} + \widehat{B}$
$\operatorname{bd}[B,C] = \{B\} \cup \{C\}$	$\partial\widehat{[B,C]} = \widehat{B} + \widehat{C}$
$\operatorname{bd}[C,D] = \{C\} \cup \{D\}$	$\partial\widehat{[C,D]} = \widehat{C} + \widehat{D}$
$\operatorname{bd}[D,A] = \{D\} \cup \{A\}$	$\partial\widehat{[D,A]} = \widehat{D} + \widehat{A}$

Table 1.3. Topology and algebra of boundaries in Q.

Since $\Gamma^1 \subset Q$, it is not surprising to see the contents of Table 1.2 contained in Table 1.3. Now observe that

$$\partial\widehat{Q} = \widehat{[A,B]} + \widehat{[B,C]} + \widehat{[C,D]} + \widehat{[D,A]}.$$

Equation (1.1) indicated that the cycle $\widehat{[A,B]} + \widehat{[B,C]} + \widehat{[C,D]} + \widehat{[D,A]}$ was the interesting algebraic aspect of Γ^1. In Q it appears as the boundary of an object. The observation we will make is that *cycles which are boundaries are uninteresting and should be ignored.*

Restating this purely algebraically, what we are looking for are things that get mapped to zero, i.e. elements of the *kernel* of the boundary operator. Furthermore, if this cycle is a *boundary*, i.e. the image of the boundary operator, then we wish to ignore it. From an algebraic point of view this means we want, somehow, to set boundaries equal to zero.

The reader may have wondered why after introducing the notion of a loop we suddenly switched to the language of cycles. Loops are combinatorial objects, i.e. lists that our computer can store. Cycles on the other hand are algebraic objects. The comments of the previous paragraph have no simple conceptual correspondence on the combinatorial level.

We have by now introduced many vague and complicated notions. If you feel things are spinning out of control - don't worry. Admittedly, there are a lot of loose ends that we need to tie up and we will begin to do so in the next chapter. The process of developing new mathematics typically involves developing new intuitions and finding new patterns - in this case we have the advantage of knowing that it will all work out in the end. For now let's just enjoy trying to match topology and algebra.

Exercises

1.4 Repeat the discussion of this section using Figures 1.1(c) and (d). In particular, make up a topology and algebra table and identify the cycles.

1.5 Repeat the above computations for a graph which represents a triangle in the plane.

1.3 Keeping Track of Directions

We want to repeat the discussion of the last section, but this time we will try to avoid counting the algebraic objects modulo 2. To do this we will consider I and Γ^1 as the explicit subsets of $\mathbf{R}^2$ indicated in Figure 1.6. Of course, this figure looks a lot like Figure 1.4. However, we have added arrows which we can think of as the standard directions of the x and y-axis.

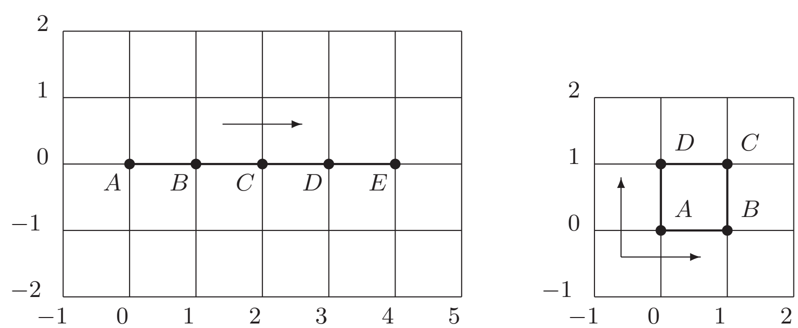


Fig. 1.6. The sets I and Γ^1 as explicit subsets of $\mathbf{R}^2$.

We will use these arrows to give a sense of direction to the edges and use the corresponding algebra to keep track of this as follows. Consider the edge $[A, B]$ in either I or Γ^1. On the combinatorial level it is defined by its endpoints A and B. Thus, in principle we could denote the edge by $[A, B]$ or $[B, A]$. For the computations performed in the previous subsection, this would make no difference (check it). Now, however, we want to distinguish these two cases. In particular, since the x-coordinate value of the vertex A is less than that of B we insist on writing the edge as $[A, B]$. Consider the edge with vertices C and D in I and Γ^1. In the first case we write $[C, D]$, but in the latter case we write $[D, C]$. In a similar vein, the edge with vertices A and D in Γ^1 is written as $[A, D]$ since the y-coordinate of A is less than the y-coordinate of D.

Having insisted on a specific order to denote the edges, we should not lose track of this when we apply the boundary operator. Let us declare that

$$\partial\widehat{[A, B]} := \widehat{B} - \widehat{A}$$

as a way to keep track of the fact that vertex a comes before the vertex B. Of course, we will still insist that ∂ be linear.

Using this linearity on the algebra generated by the edges of I we obtain

$$\begin{aligned}\partial\left(\widehat{[A, B]} + \widehat{[B, C]} + \widehat{[C, D]} + \widehat{[D, E]}\right) &= \widehat{B} - \widehat{A} + \widehat{C} - \widehat{B} + \widehat{D} - \widehat{C} + \widehat{E} - \widehat{D} \\ &= \widehat{E} - \widehat{A}.\end{aligned}$$

Again, we see that there is consistency between the algebra and the topology, since if we think of I as a single edge, then $\operatorname{bd} I = \{E\} \cup \{A\}$ and the minus sign suggests traversing from A to E.

Applying the boundary operator to the algebraic objects generated by the edges of Γ^1 gives rise to Table 1.4.

Topology		Algebra
$\operatorname{bd}[A, B] = \{A\} \cup \{B\}$		$\partial\widehat{[A, B]} = \widehat{B} - \widehat{A}$
$\operatorname{bd}[B, C] = \{B\} \cup \{C\}$		$\partial\widehat{[B, C]} = \widehat{C} - \widehat{B}$
$\operatorname{bd}[D, C] = \{C\} \cup \{D\}$		$\partial\widehat{[D, C]} = \widehat{C} - \widehat{D}$
$\operatorname{bd}[A, D] = \{D\} \cup \{A\}$		$\partial\widehat{[A, D]} = \widehat{D} - \widehat{A}$

Table 1.4. Topology and algebra of boundaries in Γ^1 remembering the directions of the x and y-axis.

Go around Γ^1 in a counterclockwise direction keeping track of the directions through which you traverse the edges. Now compare this with the directions of the edges defined by the x and y axis. Observe that we are traversing the edges $[D, C]$ and $[A, D]$ in the opposite order from the orientations of the x and y-axis. To keep track of this on the algebraic level we will write

$$[\widehat{A,B}] + [\widehat{B,C}] - [\widehat{D,C}] - [\widehat{A,D}].$$

Notice that we see a significant difference with the purely topological representation

$$\Gamma^1 = [A,B] \cup [B,C] \cup [D,C] \cup [A,D].$$

However,

$$\begin{aligned}\partial\left([\widehat{A,B}] + [\widehat{B,C}] - [\widehat{D,C}] - [\widehat{A,D}]\right) &= \partial\left([\widehat{A,B}]\right) + \partial\left([\widehat{B,C}]\right) \\ &\quad - \partial\left([\widehat{D,C}]\right) - \partial\left([\widehat{A,D}]\right) \\ &= \widehat{B} - \widehat{A} + \widehat{C} - \widehat{B} + \widehat{D} - \widehat{C} + \widehat{A} - \widehat{D} \\ &= 0.\end{aligned}$$

So again, we see that the algebra that corresponds to the interesting topology is a cycle - a sum of algebraic objects whose boundaries add up to zero.

We still need to understand what happens to this algebra when we fill in Γ^1 by Q. Consider Figure 1.7. Table 1.5 contains the topological boundary information and algebra that we are associating to it for C^2.

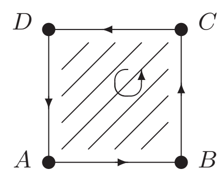


Fig. 1.7. The set Q and associated directions.

Since $\Gamma^1 \subset Q$, we again see the contents of Table 1.4 contained in Table 1.5. Keeping track of directions and walking around the edge of Q in a counterclockwise direction it seems reasonable to define

$$\partial\widehat{Q} = [\widehat{A,B}] + [\widehat{B,C}] - [\widehat{D,C}[- [\widehat{A,D}].$$

Equation (1.1) indicated that the cycle $[\widehat{A,B}]+[\widehat{B,C}]-[\widehat{D,C}[-[\widehat{A,D}]$ was the interesting algebraic aspect of Γ^1. In Q it appears as the boundary of an object. Again, the observation that we will make is: cycles which are boundaries should be considered uninteresting.

Exercises

1.6 Repeat the discussion of this section using Figures 1.1(c) and (d). In particular, make up a topology and algebra table and identify the cycles.

Topology	Algebra
$\operatorname{bd} Q = \Gamma^1 = [A,B] \cup [B,C] \cup [C,D] \cup [D,A]$	$\partial\widehat{Q} = \widehat{[A,B]} + \widehat{[B,C]} + \widehat{[C,D]} + \widehat{[D,A]}$
$\operatorname{bd}[A,b] = \{A\} \cup \{B\}$ $\operatorname{bd}[B,C] = \{B\} \cup \{C\}$ $\operatorname{bd}[C,D] = \{C\} \cup \{D\}$ $\operatorname{bd}[D,A] = \{D\} \cup \{A\}$	$\partial\widehat{[A,B]} = \widehat{B} - \widehat{A}$ $\partial\widehat{[B,C]} = \widehat{C} - \widehat{B}$ $\partial\widehat{[C,D]} = \widehat{D} - \widehat{C}$ $\partial\widehat{[D,A]} = \widehat{A} - \widehat{D}$

Table 1.5. Topology and algebra of boundaries in C^2.

1.4 Mod 2 Homology of Graphs

We have done the same example twice using different types of arithmetic but the conclusion was the same. We should look for a linear operator that somehow algebraically mimics what is done by taking the edge or boundary of a set. Then, having found this operator we should look for cycles (elements of the kernel), but ignore boundaries (elements of the image). This is still pretty fuzzy so lets do it again; a little slower and more formally, but in the general setting of graphs and using linear algebra. We are not yet aiming for rigor, but we do want to suggest that there is a way to deal with all the ideas that are being thrown about in a systematic manner.

The linear algebra we are going to do, *mod 2 linear algebra*, may seem a little strange at first, but hopefully you will recognize that all the important ideas, such as vector addition, basis, matrices, etc. still hold. Furthermore, it has the nice property that the number of algebraic elements is very small, so we can explicitly write them out.

Let G be a graph with a fixed representation as a combinatorial graph with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We will construct two vector spaces $C_0(G;\mathbf{Z}_2)$ and $C_1(G;\mathbf{Z}_2)$ as follows. Declare the set of vertices $\mathcal{V}$ to be the set of basis elements of $C_0(G;\mathbf{Z}_2)$. Thus, if $\mathcal{V} = \{v_1,\dots,v_n\}$, then the collection

$$\{\widehat{v}_i \mid i = 1,\dots,n\}$$

is a basis for $C_0(G;\mathbf{Z}_2)$. Notice that we are continuing to use the hat notation to distinguish between an algebraic object (a basis element) and a combinatorial object (an element in a list).[1] Since $\{\widehat{v}_i \mid i = 1,\dots,n\}$ is a basis, elements of $C_0(G;\mathbf{Z}_2)$ take the form

[1] Since we are trying to be a little more formal in this section, perhaps this is a good place to emphasize the fact that elements of $\mathcal{V}$ are really combinatorial objects. These are the objects that we want the computer to manipulate, therefore they are just elements of a list stored in the computer. Of course, since we are really interested in understanding the topology of the graph G, we make the

$$c = \alpha_1 \widehat{v}_1 + \alpha_2 \widehat{v}_2 + \cdots + \alpha_n \widehat{v}_n. \tag{1.2}$$

Example 1.4 Let us assume that $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$. Then the basis elements for $C_0(G; \mathbf{Z}_2)$ are $\{\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4\}$. Therefore, every element of $C_0(G; \mathbf{Z}_2)$ can be written as

$$c = \alpha_1 \widehat{v}_1 + \alpha_2 \widehat{v}_2 + \alpha_3 \widehat{v}_3 + \alpha_4 \widehat{v}_4.$$

However, since $\alpha_i \in \{0, 1\}$ we can write out all the elements of $C_0(G; \mathbf{Z}_2)$. In particular,

$$C_0(G; \mathbf{Z}_2) = \left\{ \begin{array}{c} 0,\ \widehat{v}_1,\ \widehat{v}_2,\ \widehat{v}_3,\ \widehat{v}_4, \\ \widehat{v}_1 + \widehat{v}_2,\ \widehat{v}_1 + \widehat{v}_3,\ \widehat{v}_1 + \widehat{v}_4,\ \widehat{v}_2 + \widehat{v}_3,\ \widehat{v}_2 + \widehat{v}_4,\ \widehat{v}_3 + \widehat{v}_4, \\ \widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3,\ \widehat{v}_1 + \widehat{v}_2 + \widehat{v}_4,\ \widehat{v}_1 + \widehat{v}_3 + \widehat{v}_4,\ \widehat{v}_2 + \widehat{v}_3 + \widehat{v}_4, \\ \widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3 + \widehat{v}_4 \end{array} \right\}$$

Each element of $C_0(G; \mathbf{Z}_2)$ is a vector and, of course we are allowed to add vectors, but mod 2, e.g.

$$(\widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3) + (\widehat{v}_1 + \widehat{v}_2 + \widehat{v}_4) = 2\widehat{v}_1 + 2\widehat{v}_2 + \widehat{v}_3 + \widehat{v}_4 = 0 + 0 + \widehat{v}_3 + \widehat{v}_4 = \widehat{v}_3 + \widehat{v}_4.$$

The symbol $\mathbf{Z}_2$ is used to remind us that we are adding mod 2.

Returning to the general discussion, let the set of edges $\mathcal{E}$ be the set of basis elements of $C_1(G; \mathbf{Z}_2)$. If $\mathcal{E} = \{e_1, \ldots, e_k\}$, then the collection $\{\widehat{e}_i \mid i = 1, \ldots, k\}$ is a basis for $C_1(G; \mathbf{Z}_2)$ and an element of $C_1(G; \mathbf{Z}_2)$ takes the form

$$c = \alpha_1 \widehat{e}_1 + \alpha_2 \widehat{e}_2 + \cdots + \alpha_k \widehat{e}_k$$

where again α_i is 0 or 1. The vector spaces $C_k(G; \mathbf{Z}_2)$ are called the *k-chains* for G.

It is convenient to introduce two more vector spaces $C_2(G; \mathbf{Z}_2)$ and $C_{-1}(G; \mathbf{Z}_2)$. We will always take $C_{-1}(G; \mathbf{Z}_2)$ to be the trivial vector space $\mathbf{0}$, i.e. the vector space consisting of exactly one element 0. For graphs we will also set $C_2(G; \mathbf{Z}_2)$ to be the trivial vector space. As we will see in Chapter 2 for more complicated spaces this need not be the case.

We now need to formally define the boundary operators that were alluded to earlier. Let

$$\begin{aligned} \partial_0 &: C_0(G; \mathbf{Z}_2) \to C_{-1}(G; \mathbf{Z}_2) \\ \partial_1 &: C_1(G; \mathbf{Z}_2) \to C_0(G; \mathbf{Z}_2) \\ \partial_2 &: C_2(G; \mathbf{Z}_2) \to C_1(G; \mathbf{Z}_2) \end{aligned}$$

be *linear maps*. Since $C_{-1}(G; \mathbf{Z}_2) = \mathbf{0}$, it is clear that the image of ∂_0 must be zero. For similar reasons, the same must be true for ∂_2. Since we have chosen

identification of $v \in \mathcal{V}$ with $v \in G \subset \mathbf{R}^3$. But it is us who make this identification, not the computer. Furthermore, as will become clear especially in Part II where we discuss the applications of homology, the ability to make this identification, and thereby, pass from combinatorics to topology is a very powerful technique.

bases for the vector spaces $C_1(G;\mathbf{Z}_2)$ and $C_0(G;\mathbf{Z}_2)$, we can express ∂_1 as a matrix. The entries of ∂_1 are determined by how ∂_1 acts on the basis elements, i.e. the edges $\mathbf{e}_i$. In line with the previous discussion we make the following definition. Let the edge e_i have vertices v_j and v_k. Define

$$\partial_1 \widehat{e}_i := \widehat{v}_j + \widehat{v}_k.$$

In our earlier example we were interested in cycles, i.e. objects that get mapped to 0 by ∂. In general, given a linear map A, the set of elements that get sent to 0 is called the *kernel* of A and is denoted by $\ker A$. Thus, the set of cycles forms the kernel of ∂. Because the set of cycles plays such an important role it has its own notation:

$$Z_0(G;\mathbf{Z}_2) := \ker \partial_0 = \{c \in C_0(G;\mathbf{Z}_2) \mid \partial_0 c = 0\}$$
$$Z_1(G;\mathbf{Z}_2) := \ker \partial_1 = \{c \in C_1(G;\mathbf{Z}_2) \mid \partial_1 c = 0\}$$

Since $C_{-1}(G;\mathbf{Z}_2) = 0$ it is obvious that $Z_0(G;\mathbf{Z}_2) = C_0(G;\mathbf{Z}_2)$, i.e. everything in $C_0(G;\mathbf{Z}_2)$ gets sent to 0.

We also observed that cycles which are boundaries are not interesting. To formally state this, define the set of boundaries to be the image of the boundary operator. More precisely,

$$B_0(G;\mathbf{Z}_2) := \operatorname{im} \partial_1 = \{b \in C_0(G;\mathbf{Z}_2) \mid \exists\, c \in C_1(G;\mathbf{Z}_2) \text{ such that } \partial_1 c = b\}$$
$$B_1(G;\mathbf{Z}_2) := \operatorname{im} \partial_2 = \{b \in C_1(G;\mathbf{Z}_2) \mid \exists\, c \in C_2(G;\mathbf{Z}_2) \text{ such that } \partial_2 c = b\}$$

Recall that we set $C_2(G;\mathbf{Z}_2) = \mathbf{0}$, thus $B_1(G;\mathbf{Z}_2) = \mathbf{0}$.

Observe that $B_0(G;\mathbf{Z}_2) \subset C_0(G;\mathbf{Z}_2) = Z_0(G;\mathbf{Z}_2)$, i.e. every 0-boundary is a 0-cycle. We shall show later that every boundary is a cycle, i.e.

$$B_k(G;\mathbf{Z}_2) \subset Z_k(G;\mathbf{Z}_2).$$

This is a very important fact - but not at all obvious at this point.

We can finally define homology in this rather special setting. For $k = 0, 1$ the k-th homology with $\mathbf{Z}_2$ coefficients is defined to be the quotient space

$$H_k(G;\mathbf{Z}_2) := Z_k(G;\mathbf{Z}_2)/B_k(G;\mathbf{Z}_2).$$

If you have not worked with quotient spaces, then the obvious question is: *What does this notation mean?*

Let us step back for a moment and remember what we are trying to do. Recall that the interesting objects are cycles, but if a cycle is a boundary then it is no longer interesting. Thus, we begin with a cycle $z \in Z_k(G;\mathbf{Z}_2)$. It is possible that $z \in B_k(G;\mathbf{Z}_2)$. In this case we want z to be uninteresting. From an algebraic point of view we can take this to mean that we want to set z equal to 0.

Now consider two cycles $z_1, z_2 \in Z_i(G;\mathbf{Z}_2)$. What if there exists a boundary $b \in B_k(G;\mathbf{Z}_2)$ such that

$$z_1 + b = z_2?$$

Since boundaries are supposed to be 0, this suggests that b should be zero and hence that we want z_1 and z_2 to be the same.

Mathematically when we want different objects to be the same we form *equivalence classes*. With this in mind we define an equivalence class on the set of cycles by

$$z_1 \sim z_2 \quad \text{if and only if} \quad z_1 + b = z_2$$

for some $b \in B_k(G; \mathbf{Z}_2)$. The notation $\sim$ means equivalent. The equivalence class of the cycle $z \in Z_k(G; \mathbf{Z}_2)$ is the set of all cycles which are equivalent to z. It is denoted by $[z]$. Thus

$$[z] := \{z' \in Z_k(G; \mathbf{Z}_2) \mid z \sim z'\}.$$

Therefore, $z_1 \sim z_2$ is the same as saying $[z_1] = [z_2]$. The set of equivalence classes make up the elements of $H_k(G; \mathbf{Z}_2)$.

This brief discussion contains several fundamental but nontrivial mathematical concepts, so to help make some of these ideas clearer consider the following examples.

Example 1.5 Let us start with the trivial graph consisting of a single point, $G = \{v\}$. Then

$$\mathcal{V} = \{v\}, \quad \mathcal{E} = \emptyset.$$

These are used to generate the bases for the chains. In particular, the basis for $C_0(G; \mathbf{Z}_2)$ is $\{\widehat{v}\}$. This means that any element $v \in C_0(G; \mathbf{Z}_2)$ has the form

$$c = \alpha \widehat{v}.$$

If $\alpha = 0$, then $c = 0$. If $\alpha = 1$, then $c = \widehat{v}$. Thus,

$$C_0(G; \mathbf{Z}_2) = \{0, \widehat{v}\}.$$

Since by definition, $\partial_0 = 0$, it follows that $Z_0(G; \mathbf{Z}_2) = C_0(G; \mathbf{Z}_2)$. Thus,

$$Z_0(G; \mathbf{Z}_2) = \{0, \widehat{v}\}.$$

The basis for $C_1(G; \mathbf{Z}_2)$ is the empty set, therefore

$$C_1(G; \mathbf{Z}_2) = \{0\}.$$

In other words, $C_1(G; \mathbf{Z}_2) = \mathbf{0}$, the trivial vector space. By definition, $Z_1(G; \mathbf{Z}_2) \subset C_1(G; \mathbf{Z}_2)$, so $Z_1(G; \mathbf{Z}_2) = \mathbf{0}$.

Since $C_1(G; \mathbf{Z}_2) = \mathbf{0}$, the image of $C_1(G; \mathbf{Z}_2)$ under ∂_1 must also be trivial. Thus, $B_0(G; \mathbf{Z}_2) = \mathbf{0}$. Now consider two cycles $z_1, z_2 \in Z_0(G; \mathbf{Z}_2)$. $B_0(G; \mathbf{Z}_2) = \mathbf{0}$ implies that $z_1 \sim z_2$, if and only if $z_1 = z_2$. Therefore,

$$H_0(G; \mathbf{Z}_2) = Z_0(G; \mathbf{Z}_2) = \{[0], [\widehat{v}]\}.$$

Of course, since $Z_1(G; \mathbf{Z}_2) = \mathbf{0}$, it follows that

$$H_1(G; \mathbf{Z}_2) = \mathbf{0}.$$

Example 1.6 Let G be the graph of Figure 1.4 representing I. Then,

$$\mathcal{V} = \{A, B, C, D, E\}$$
$$\mathcal{E} = \{[A, B], [B, C][C, D], [D, E]\}$$

Thus, the basis for the 0-chains, $C_0(G; \mathbf{Z}_2)$, is

$$\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}, \widehat{E}\}$$

while the basis for the 1-chains, $C_1(G; \mathbf{Z}_2)$, is

$$\{\widehat{[A, B]}, \widehat{[B, C]}, \widehat{[C, D]}, \widehat{[D, E]}\}.$$

Unlike the previous example, we really need to write down the boundary operator $\partial_1 : C_1(G; \mathbf{Z}_2) \to C_0(G; \mathbf{Z}_2)$. Given the above mentioned bases, since ∂_1 is a linear map it can be written as a matrix. To do this it is convenient to use the notation of column vectors.[2] So let

$$\widehat{A} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \widehat{E} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\widehat{[A, B]} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{[B, C]} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{[C, D]} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \widehat{[D, E]} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

With this convention, ∂_1 becomes the 5×4 matrix

$$\partial_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lets do a quick check. For example

$$\partial_1 \widehat{[B, C]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \widehat{B} + \widehat{C}.$$

[2] If you are surprised that our vectors are five dimensional and four dimensional, observe that we are talking about elements of the vector spaces $C_k(G; \mathbf{Z}_2)$. This is a *different* vector space from $\mathbf{R}^2$ in which we see the vertices and edges as points and line segments.

The next step is to compute the cycles, i.e. to find $Z_1(G; \mathbf{Z}_2) := \ker \partial_1$. Observe that by definition the chain $c \in C_1(G; \mathbf{Z}_2)$ is in $Z_1(G; \mathbf{Z}_2)$ if and only if $\partial_1 c = \widehat{0}$. Again, since $c \in C_1(G; \mathbf{Z}_2)$, it can be written as a sum of basis elements, i.e.

$$c = \alpha_1 [\widehat{A, B}] + \alpha_2 [\widehat{B, C}] + \alpha_3 [\widehat{C, D}] + \alpha_4 [\widehat{D, E}].$$

Writing this in the form of a column vector we have

$$c = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}.$$

In this form, finding $c \in \ker \partial_1$ is equivalent to solving the equation

$$\partial_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1\,0\,0\,0 \\ 1\,1\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \\ \alpha_2 + \alpha_3 \\ \alpha_3 + \alpha_4 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that $\alpha_i = 0$ for $i = 1, \ldots 4$. Thus, the only element in $Z_1(G; \mathbf{Z}_2)$ is the zero vector and hence $Z_1(G; \mathbf{Z}_2) = \{0\}$. Since $\partial_2 = 0$, we have $B_1(G; \mathbf{Z}_2) = 0$. So

$$H_1(G; \mathbf{Z}_2) := Z_1(G; \mathbf{Z}_2)/B_1(G; \mathbf{Z}_2) = \{[0]\}.$$

Computing $H_0(G; \mathbf{Z}_2)$ is more interesting. We know that $Z_0(G; \mathbf{Z}_2) = C_0(G; \mathbf{Z}_2)$. So a basis for $Z_0(G; \mathbf{Z}_2)$ consists of

$$\widehat{A} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \widehat{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \widehat{E} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

What about $B_0(G; \mathbf{Z}_2)$? This is the image of ∂_1 which is spanned by the images of each of the basis elements. This we can compute. In particular,

$$\partial_1 [\widehat{A, B}] = \begin{bmatrix} 1\,0\,0\,0 \\ 1\,1\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly,

$$\partial_1[\widehat{B,C}] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \partial_1[\widehat{C,D}] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \partial_1[\widehat{D,E}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

From this we can conclude that if $b \in B_0(G; \mathbf{Z}_2)$, then

$$b = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

where $\alpha_i \in \{0, 1\}$.

It is easy to check that no basis element of $Z_0(G; \mathbf{Z}_2)$ is in $B_0(G; \mathbf{Z}_2)$. For example, if for some $\alpha_1, \alpha_2, \alpha_3, , \alpha_4, \alpha_5 \in \{0, 1\}$ we have

$$\widehat{A} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

then

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ \alpha_3 + \alpha_4 &= 0 \\ \alpha_4 &= 0 \end{aligned}$$

Since we count modulo 2, we conclude from the second equation that $\alpha_2 = 1$. But then similarly also $\alpha_3 = \alpha_4 = 1$, which contradicts the last equation. This leads to the conclusion that the equivalence class

$$[\widehat{A}] \neq [0] \in H_0(G; \mathbf{Z}_2).$$

On the other hand, again counting mod 2

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can rewrite this equation as

$$\widehat{A} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \widehat{B}.$$

Since

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in B_0(G; \mathbf{Z}_2)$$

this implies that

$$\widehat{A} \sim \widehat{B}.$$

Therefore, on the level of homology

$$[\widehat{A}] = [\widehat{B}] \in H_0(G; \mathbf{Z}_2).$$

We leave it to the reader to check that, in fact,

$$[\widehat{A}] = [\widehat{B}] = [\widehat{C}] = [\widehat{D}] = [\widehat{E}] \in H_0(G; \mathbf{Z}_2). \tag{1.3}$$

Thus, if we write out the elements of $H_0(G; \mathbf{Z}_2)$ without repetition we get

$$\{[0], [\widehat{A}]\} \subset H_0(G; \mathbf{Z}_2). \tag{1.4}$$

The question is if we have found all homology classes in $H_0(G; \mathbf{Z}_2)$. To answer it we need to understand that the vector space $Z_0(G; \mathbf{Z}_2)$ *induces* the structure of a vector space on $H_0(G; \mathbf{Z}_2)$. For this end consider the two cycles $\widehat{A}, \widehat{B}$. They both induce the same element in $H_0(G; \mathbf{Z}_2)$ because $[\widehat{A}] = [\widehat{B}]$. Since $Z_0(G; \mathbf{Z}_2)$ is a vector space we can add $\widehat{A}$ and $\widehat{B}$ to get $\widehat{A} + \widehat{B}$. Observe that in column notation

$$\widehat{A} + \widehat{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in B_0(G; \mathbf{Z}_2).$$

Let us collect all this information in the context of the equivalence classes.

$$[0] \stackrel{1}{=} 2[\widehat{A}] = [\widehat{A}] + [\widehat{A}] \stackrel{2}{=} [\widehat{A}] + [\widehat{B}] \stackrel{3}{=} [\widehat{A} + \widehat{B}] \stackrel{4}{=} [0].$$

Equality 1 follows from the fact that we are still doing mod 2 arithmetic. Equality 2 holds because $\widehat{A} \sim \widehat{B}$. Equality 3 is what we mean by saying that $Z_0(G; \mathbf{Z}_2)$ induces the structure of a vector space on $H_0(G; \mathbf{Z}_2)$. More precisely, to add equivalence classes we just add representatives of the equivalence classes.[3] The last equality follows from the fact that $\widehat{A}+\widehat{B} \in B_0(G;\mathbf{Z}_2)$.

Equipped with the vector space structure of $H_0(G;\mathbf{Z}_2)$ we easily see that the homology class of an arbitrary cycle is

$$[\alpha_1\widehat{A}+\alpha_2\widehat{B}+\alpha_3\widehat{C}+\alpha_4\widehat{D}+\alpha_5\widehat{E}] = (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)[\widehat{A}] \in \{[0],[\widehat{A}]\}.$$

Therefore

$$H_0(G;\mathbf{Z}_2) = \{[0],[\widehat{A}]\}. \tag{1.5}$$

A final comment is needed before ending this example. We motivated computing homology, by arguing that we wanted an algebraic method for determining whether pictures have bounded regions or not. We introduced the combinatorial graphs as a combinatorial representation that was then turned into algebra. We do not want the final answer to depend on the graph. We also mentioned that proving this is difficult. But to emphasize this we will write the homology in terms of the set rather than the graph. In other words what we claim (without proof!) is that

$$H_1(I;\mathbf{Z}_2) = \mathbf{0} \quad \text{and} \quad H_0(I;\mathbf{Z}_2) = \{[0],[\widehat{A}]\}.$$

Notice, however, that the chains depend explicitly on the combinatorial graph, therefore it makes no sense to write $C_k(I;\mathbf{Z}_2)$.

Example 1.7 Let G be the graph of Figure 1.4 representing Γ^1. Then,

$$\begin{aligned}\mathcal{V} &= \{A,B,C,D\}\\ \mathcal{E} &= \{[A,B],[B,C],[D,C],[A,D]\}.\end{aligned}$$

The computation of the homology begins just as in the previous example. The basis for the 0-chains, $C_0(G;\mathbf{Z}_2)$, is

$$\{\widehat{A},\widehat{B},\widehat{C},\widehat{D}\}$$

while the basis for the 1-chains, $C_1(G;\mathbf{Z}_2)$, is

$$\{\widehat{[A,B]},\widehat{[B,C]},\widehat{[D,C]},\widehat{[A,D]}\}.$$

Again, using column vectors let

$$\widehat{A} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ \widehat{B} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\ \widehat{C} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ \widehat{D} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

and

[3] Of course it needs to be checked that this addition is well defined. This is done in Chapter 13.

$$[\widehat{A,B}] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ [\widehat{B,C}] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ [\widehat{D,C}] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ [\widehat{A,D}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

With this convention, ∂_1 becomes the 4×4 matrix

$$\partial_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

To compute $Z_1(G; \mathbf{Z}_2) := \ker \partial_1$, we need to solve the equation

$$\partial_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_4 \\ \alpha_1 + \alpha_2 \\ \alpha_2 + \alpha_3 \\ \alpha_3 + \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Observe that since we are using mod 2 arithmetic,

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$$

is a solution. In particular, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ is the only non-zero solution. Thus,

$$Z_1(G; \mathbf{Z}_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

As in the previous example $B_1(G; \mathbf{Z}_2) = \widehat{0}$. So

$$H_1(\Gamma^1; \mathbf{Z}_2) := Z_1(G; \mathbf{Z}_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Notice that this is different from the previous two examples and has to do with the fact that Γ^1 encloses a region in the plane.

We still need to compute $H_0(\Gamma^1; \mathbf{Z}_2)$. We know that $Z_0(G; \mathbf{Z}_2) = C_0(G; \mathbf{Z}_2)$. The next step is to understand $B_0(G; \mathbf{Z}_2)$. Again, we look at how ∂_1 acts on the basis elements of $C_1(G; \mathbf{Z}_2)$ and conclude that

$$\partial_1[\widehat{A,B}] = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \partial_1[\widehat{B,C}] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \partial_1[\widehat{D,C}] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \partial_1[\widehat{A,D}] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As before, $\widehat{A} \notin B_0(G; \mathbf{Z}_2)$, so

$$[\widehat{A}] \neq [0] \in H_0(\Gamma^1; \mathbf{Z}_2)$$

but

$$\widehat{A} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \widehat{B}$$

and so

$$\widehat{A} \sim \widehat{B}.$$

It is left to the reader to check that, as in the previous example,

$$H_0(\Gamma^1; \mathbf{Z}_2) = \{[0], [\widehat{A}]\}.$$

Remark 1.8 Several times in this section we have ended up with a vector space containing two elements $\{[0], [\widehat{A}]\}$. Observe that we have the following relations under vector addition with mod 2 arithmetic

$$[0] + [0] = [0], \quad [0] + [\widehat{A}] = [\widehat{A}], \quad [\widehat{A}] + [\widehat{A}] = [0].$$

We can identify this with mod 2 arithmetic, i.e. we can consider the set $\mathbf{Z}_2 := \{0, 1\}$ where

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 1 = 0.$$

From now on we will do this. In particular, this allows us to write the homology groups from the previous example as

$$H_0(\Gamma^1; \mathbf{Z}_2) \cong \mathbf{Z}_2, \quad H_1(\Gamma^1; \mathbf{Z}_2) \cong \mathbf{Z}_2$$

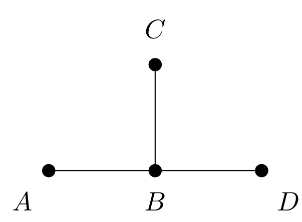
Exercises

1.7 Verify equation (1.3).

1.8 Verify equation (1.5) by explicitly writing out all the elements of $Z_0(G; \mathbf{Z}_2)$ and showing that each element is similar to either 0 or $\widehat{A}$.

1.9 Let G be the graph with edges $\mathcal{E} = \{[v_1, v_2]\}$ and vertices $\mathcal{V} = \{v_1, v_2\}$. Compute $H_k(G; \mathbf{Z}_2)$ for $k = 0, 1$.

1.10 Compute $H_k(G; \mathbf{Z}_2)$ for $k = 0, 1$, where G is the graph



1.11 Compute $H_k(G; \mathbf{Z}_2)$ for $k = 0, 1$, where G is the graph with vertices $\{v_1, v_2, v_3\}$ and edges $\{[v_1, v_2], [v_2, v_3], [v_3, v_1]\}$.

1.12 Let G^n be a graph with n vertices where every pair of distinct vertices is connected by one edge. Create input files for the Homology program for several values of n starting from $n = 4$. Use the Homology program to compute the dimensions r_k of $H_k(G^n; \mathbf{Z}_2)$. Make a conjecture about the formula for every $n \geq 3$.

2

Cubical Homology

In Chapter 1 we suggested what were the important elements in Homology. In particular, we used the edges and vertices of a graph to generate algebraic objects that measured the non-triviality of the topology of the graph. In this chapter we shall formally define cubical homology. However, the first step is to generalize the combinatorics of graphs to higher dimensional spaces.

There are several ways to extract combinatoric and algebra information from a set in $\mathbf{R}^d$. An approach arising naturally from tomography, numerical computations and graphics is by means of cubical grids which subdivide the space into cubes with vertices in an integer lattice. Look for example at Figure 2.1. It shows two crossing circles which are understood as smooth curves, composed of uncountably many points. But, like any picture produced by a computer, there is only a finite amount of information involved. If we zoom in a section of the figure we will see in Figure 2.2 a chain of small squares called in computer graphics *pixels*. Note that any two pixels are either disjoint or intersect at a common edge or vertex. What we see has a nice combinatorial structure and we should be able to extract algebra out of it. This approach is the Cubical Homology Theory presented here.

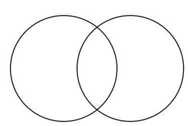


Fig. 2.1. A typical picture produced by computer.

In numerical and graphical analysis one needs to consider very fine cubical grids. The size of cubes of a grid cannot be arbitrarily small because of the

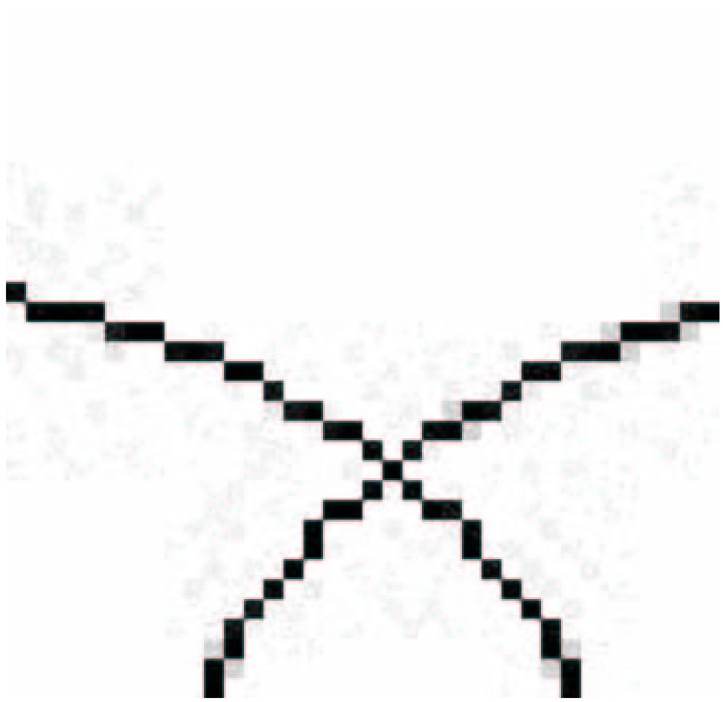


Fig. 2.2. A zoomed in section of the previous picture.

computer's capacity. From a theoretical point of view, the size of a grid is just a question of choice of units. With appropriate units we may assume in this chapter that each cube is unitary i.e. it has sides of length 1 and vertices with integer coordinates. In Chapter 6 we will investigate what happens with the algebra extracted from a cubical grid when we change units.

2.1 Cubical Sets

Given a graph we obtained a combinatorial object by defining it in terms of vertices and edges. Of course, these will not suffice to combinatorialize higher dimensional topological spaces. Thus, in this section we formally introduce the notion of cubes which form the building blocks for the homology theory presented in this book.

2.1.1 Elementary Cubes

Definition 2.1 An *elementary interval* is a closed interval $I \subset \mathbf{R}$ of the form

$$I = [l, l+1] \quad \text{or} \quad I = [l, l]$$

for some $l \in \mathbf{Z}$. To simplify the notation we will write

$$[l] = [l, l]$$

for an interval that contains only one point. Elementary intervals that consist of a single point are *degenerate*. Elementary intervals of length one are *nondegenerate*.

Example 2.2 The intervals $[2, 3]$, $[-15, -14]$, and $[7]$ are all examples of elementary intervals. On the other hand, $[\frac{1}{2}, \frac{3}{2}]$ is not an elementary interval since the boundary points are not integers. Similarly, $[1, 3]$ is not an elementary interval since the length of the interval is greater than 1.

Definition 2.3 An *elementary cube* Q is a finite product of elementary intervals, i.e.

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbf{R}^d$$

where each I_i is an elementary interval. The set of all elementary cubes in $\mathbf{R}^d$ is denoted by $\mathcal{K}^d$. The set of all elementary cubes is denoted by $\mathcal{K}$, i.e.

$$\mathcal{K} := \bigcup_{d=1}^{\infty} \mathcal{K}^d.$$

Figure 2.3 indicates a variety of elementary cubes. Observe that the cube $[1, 2] \subset \mathbf{R}$ is different from the cube $[1, 2] \times [0] \subset \mathbf{R}^2$ since they are subsets of different spaces. Of course using the inclusion map $i : \mathbf{R} \to \mathbf{R}^2$ given by $i(x) = (x, 0)$ we can identify these two elementary cubes. However, we will take great care in this book to explicitly state this identification if we make it. Thus, if the identification is not clearly stated, then they should be treated as distinct sets.

Another natural identification which one should be aware of is the correspondence between a point in $\mathbf{R}^d$ and an elementary cube all of whose elementary intervals are degenerate. For example, $(1, 0, 2) \in \mathbf{R}^3$ is a point in a topological space, whereas $[1] \times [0] \times [2]$ is an elementary cube. The former is a topological object, while the latter will be used as a combinatorial element.

Here are some other examples of elementary cubes

$$\begin{aligned}
&Q_1 := [1, 2] \times [0, 1] \times [-2, -1] \subset \mathbf{R}^3 \\
&Q_2 := [1] \times [1, 2] \times [0, 1] = \{1\} \times [1, 2] \times [0, 1] \subset \mathbf{R}^3 \\
&Q_3 := [1, 2] \times [0] \times [-1] = [1, 2] \times \{0\} \times \{-1\} \subset \mathbf{R}^3 \\
&Q_4 := [0] \times [0] \times [0] = \{(0, 0, 0)\} \in \mathbf{R}^3 \\
&Q_5 := [-1, 0] \times [3, 4] \times [6] \times [1, 2] = [-1, 0] \times [3, 4] \times \{6\} \times [1, 2] \subset \mathbf{R}^4
\end{aligned}$$

which we shall not attempt to draw.

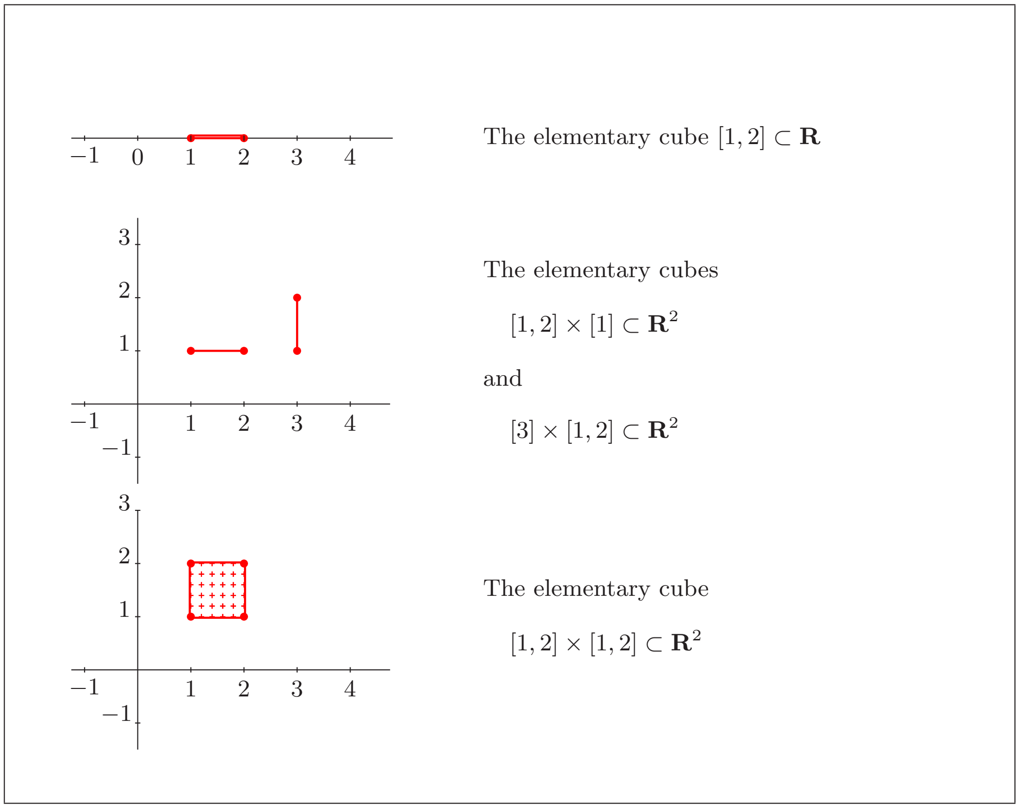


Fig. 2.3. Elementary cubes in $\mathbf{R}$ and $\mathbf{R}^2$.

Definition 2.4 Let $Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbf{R}^d$ be an elementary cube. The *embedding number* of Q is denoted by $\operatorname{emb} Q$ and is defined to be d since $Q \subset \mathbf{R}^d$. The interval I_i is referred to as the i-th *component* of Q and written as $I_i(Q)$. The *dimension* of Q is defined to be the number of nondegenerate components in Q and is denoted by $\dim Q$.

Observe that if $\operatorname{emb} Q = d$ then $Q \in \mathcal{K}^d$. Similarly, we will let

$$\mathcal{K}_k := \{Q \in \mathcal{K} \mid \dim Q = k\}$$

and

$$\mathcal{K}_k^d := \mathcal{K}_k \cap \mathcal{K}^d.$$

Example 2.5 Referring to the elementary cubes defined above we have that $I_1(Q_3) = [1, 2]$, $I_2(Q_3) = [0]$ and $I_3(Q_3) = [-1]$. Furthermore,

$$\begin{aligned}
\operatorname{emb} Q_1 &= 3 \text{ and } \dim Q_1 = 3 \\
\operatorname{emb} Q_2 &= 3 \text{ and } \dim Q_2 = 2 \\
\operatorname{emb} Q_3 &= 3 \text{ and } \dim Q_3 = 1
\end{aligned}$$

$$\operatorname{emb} Q_4 = 3 \text{ and } \dim Q_4 = 0$$
$$\operatorname{emb} Q_5 = 4 \text{ and } \dim Q_5 = 3$$

In particular, the reader should observe that the only general relation between the embedding number and the dimension of an elementary cube Q is that

$$0 \leq \dim Q \leq \operatorname{emb} Q. \tag{2.1}$$

Proposition 2.6 *Let $Q \in \mathcal{K}_k^d$ and $P \in \mathcal{K}_{k'}^{d'}$, then*

$$Q \times P \in \mathcal{K}_{k+k'}^{d+d'}.$$

Proof. Since $Q \in \mathcal{K}^d$ it can be written as the product of d elementary intervals, i.e.

$$Q = I_1 \times I_2 \times \ldots \times I_d.$$

Similarly,

$$P = J_1 \times J_2 \times \ldots \times J_{d'}$$

where each J_i is an elementary interval. Hence,

$$Q \times P = I_1 \times I_2 \times \ldots \times I_d \times J_1 \times J_2 \times \ldots \times J_{d'}$$

which is a product of $d + d'$ elementary intervals. It is left to the reader to check that $\dim(Q \times P) = \dim Q + \dim P$. □

It should be clear from the proof of Proposition 2.6 that though they lie in the same space $Q \times P \neq P \times Q$.

The following definition will allow us to decompose elementary cubes into lower dimensional objects.

Definition 2.7 Let $Q, P \in \mathcal{K}$. If $Q \subset P$, then Q is a *face* of P. This is denoted by $Q \preceq P$. If $Q \preceq P$ and $Q \neq P$, then Q is a *proper face* of P which is written as $Q \prec P$. Q is a *primary* face of P if Q is a face of P and $\dim Q = \dim P - 1$.

Example 2.8 Let $Q = [1,2] \times [1]$ and $P = [1,2] \times [1,2]$. Then $Q \prec P$ and Q is a primary face of P.

2.1.2 Cubical Sets

As was mentioned before elementary cubes make up the basic building blocks for our homology theory. Thus, to begin with, we need to restrict our attention to the following special class of topological spaces. In Chapter 11 we will consider more general spaces.

Definition 2.9 A set $X \subset \mathbf{R}^d$ is *cubical* if X can be written as a finite union of elementary cubes.

If $X \subset \mathbf{R}^d$ is a cubical set, then we adopt the following notation.

$$\mathcal{K}(X) := \{Q \in \mathcal{K} \mid Q \subset X\}$$

and

$$\mathcal{K}_k(X) := \{Q \in \mathcal{K}(X) \mid \dim Q = k\}.$$

Observe that if $Q \subset X$ and $Q \in \mathcal{K}$ then $\operatorname{emb} Q = d$, since $X \subset \mathbf{R}^d$. This in turn implies that $Q \in \mathcal{K}^d$ so to use the notation $\mathcal{K}^d(X)$ is somewhat redundant, but it serves to remind us that $X \subset \mathbf{R}^d$. Therefore, when it is convenient we will write $\mathcal{K}_k^d(X)$. In analogy with graphs, the elements of $\mathcal{K}_0(X)$ are the *vertices* of X and the elements of $\mathcal{K}_1(X)$ are the *edges* of X. More generally, the elements of $\mathcal{K}_k(X)$ are the *k-cubes* of X.

Example 2.10 Consider the set $X = [0,1] \times [0,1] \times [0,1] \subset \mathbf{R}^3$. This is an elementary cube, and hence, is a cubical set. It is easy to check that

$$\begin{aligned}
\mathcal{K}_3(X) &= \{[0,1] \times [0,1] \times [0,1]\} \\
\mathcal{K}_2(X) &= \{[0] \times [0,1] \times [0,1], [1] \times [0,1] \times [0,1], \\
&\qquad [0,1] \times [0] \times [0,1], [0,1] \times [1] \times [0,1], \\
&\qquad [0,1] \times [0,1] \times [0], [0,1] \times [0,1] \times [1]\} \\
\mathcal{K}_1(X) &= \{[0] \times [0] \times [0,1], [0] \times [1] \times [0,1], \\
&\qquad [0] \times [0,1] \times [0], [0] \times [0,1] \times [1], \\
&\qquad [1] \times [0] \times [0,1], [1] \times [1] \times [0,1], \\
&\qquad [1] \times [0,1] \times [0], [1] \times [0,1] \times [1], \\
&\qquad [0,1] \times [0] \times [0], [0,1] \times [0] \times [1], \\
&\qquad [0,1] \times [1] \times [0], [0,1] \times [1] \times [1]\} \\
\mathcal{K}_0(X) &= \{[0] \times [0] \times [0], [0] \times [0] \times [1], \\
&\qquad [0] \times [1] \times [0], [0] \times [1] \times [1], \\
&\qquad [1] \times [0] \times [0], [1] \times [0] \times [1], \\
&\qquad [1] \times [1] \times [0], [1] \times [1] \times [1]\}.
\end{aligned}$$

Example 2.11 It should be noted that the definition of a cubical set is extremely restrictive. For example, the unit circle $x^2 + y^2 = 1$ is not a cubical set. In fact, even a simple set such as a point may or may not be a cubical set. In particular, consider the set consisting of one point $P = \{(x, y, z)\} \subset \mathbf{R}^3$. P is a cubical set if and only if x, y, and z are all integers.

The following result uses some simple point set topology (see Chapter 12).

Proposition 2.12 *If $X \subset \mathbf{R}^d$ is cubical, then X is closed and bounded.*

Proof. By definition a cubical set is the finite union of elementary cubes. By Exercise 12.13 an elementary cube is closed and by Theorem 12.18 the finite union of closed sets is closed.

To show that X is bounded, let $Q \in \mathcal{K}(X)$. Then $Q = I_1 \times I_2 \times \cdots \times I_d$ where $I_i = [l_i]$ or $I_i = [l_i, l_i + 1]$. Let

$$\rho(Q) = \max_{i=1,\dots d} \{|l_i| + 1\}$$

Now set $R = \max_{Q \in \mathcal{K}(X)} \rho(Q)$. Then $X \subset B_0(0, R)$. □

2.1.3 Elementary Cells

Elementary cubes are the building blocks for the homology theory that we are developing. However for technical reasons it is useful to have additional sets to work with. For this reason we introduce the notion of elementary cells.

Definition 2.13 Let I be an elementary interval. The associated *elementary cell* is

$$\overset{\circ}{I} := \begin{cases} (l, l+1) & \text{if } I = [l, l+1], \\ [l] & \text{if } I = [l, l]. \end{cases}$$

We extend this definition to a general elementary cube $Q = I_1 \times I_2 \times \ldots \times I_d \subset \mathbf{R}^d$ by defining the associated *elementary cell* as

$$\overset{\circ}{Q} := \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d.$$

Example 2.14 Consider the elementary cube $Q = [1, 2] \times [3] \in \mathcal{K}^2_1$. The associated elementary cell is $\overset{\circ}{Q} = (1, 2) \times [3] \subset \mathbf{R}^2$.

Given a point in $\mathbf{R}^d$ we will need to be able to describe the elementary cell or cube which contains it. For this, the following two functions are useful. Let $x \in \mathbf{R}$,

$$\begin{aligned} \operatorname{floor}(x) &:= \max\{n \in \mathbf{Z} \mid n \leq x\} \\ \operatorname{ceil}(x) &:= \min\{n \in \mathbf{Z} \mid x \leq n\}. \end{aligned}$$

Some attributes of elementary cells are summarized in the following proposition.

Proposition 2.15 *Elementary cells have the following properties.*

(i) $\mathbf{R}^d = \bigcup \{\overset{\circ}{Q} \mid Q \in \mathcal{K}^d\}$.

(ii) $A \subset \mathbf{R}^d$ *bounded implies that* $\operatorname{card}\{Q \in \mathcal{K}^d \mid \overset{\circ}{Q} \cap A \neq \emptyset\} < \infty$.

(iii) If $P, Q \in \mathcal{K}$, *then* $\overset{\circ}{P} \cap \overset{\circ}{Q} = \emptyset$ *or* $P = Q$.

(iv) For every $Q \in \mathcal{K}$, $\operatorname{cl} \overset{\circ}{Q} = Q$.

(v) $Q \in \mathcal{K}^d$ *implies that* $Q = \bigcup \{\overset{\circ}{P} \mid P \in \mathcal{K}^d \text{ such that } \overset{\circ}{P} \subset Q\}$.

(vi) If X is a cubical set and $\overset{\circ}{Q} \cap X \neq \emptyset$ for some elementary cube Q, then $Q \subset X$.

Proof. (i) Obviously $\bigcup\{\overset{\circ}{Q} \mid Q \in \mathcal{K}^d\} \subset \mathbf{R}^d$. To prove the opposite inclusion take an $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$ and put

$$I_i := [\text{floor}\,(x_i), \text{ceil}\,(x_i)]$$

Then $\overset{\circ}{Q} := \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d$ is an elementary cell and $x \in \overset{\circ}{Q}$. This proves (i).

(ii) The proof is straightforward.

(iii) For elementary cubes of dimension zero and one the result is obvious. Also, it extends immediately to elementary cubes of dimension greater than one, because the intersection of products of intervals is the product of the intersections of the corresponding intervals.

(iv) See Definition 12.19 and Exercise 12.12.

(v) Consider $Q = I_1 \times I_2 \times \ldots \times I_d$ and let $x = (x_1, x_2, \ldots, x_d) \in Q$. Define

$$J_i := \begin{cases} [x_i, x_i] & \text{if } x_i \text{ is an endpoint of } I_i, \\ I_i & \text{otherwise.} \end{cases}$$

and put $P := J_1 \times J_2 \times \ldots \times J_d$. Then obviously $x \in \overset{\circ}{P}$ and $\overset{\circ}{P} \subset Q$. Hence x belongs to the right-hand-side of (v).

(vi) Let $X = P_1 \cup P_2 \cup \ldots \cup P_m$, where the P_i are elementary cubes. Then $\overset{\circ}{Q} \cap P_i \neq \emptyset$ for some $i = 1, 2, \ldots m$. It follows from (v) and (iii) that $\overset{\circ}{Q} \subset P_i$ and from (iv) that $Q \subset \text{cl}\, P_i = P_i \subset X$. □

Exercises

2.1 The *unitary d-cube* is $[0,1]^d \subset \mathbf{R}^d$. This obviously is an elementary cube.

(a) List the faces contained in $[0,1]^2$.
(b) How many primary faces are in $[0,1]^d$?
(c) How many faces of all dimensions does $[0,1]^d$ contain?

2.2 Show that any $(k-1)$-dimensional face of a $(k+1)$-dimensional elementary cube Q is a common face of exactly two k-dimensional faces of Q.

The next two exercises are a practice of using the program CubTop from the folder CubTop. The reader should first consult the README file in CubTop.

2.3 Open the file exC2d.txt in the folder CubTop/Examples of a set X presented there to see how is it constructed. You will see a list of elementary cubes in $\mathbf{R}^2$ there. Then perform the following functions.

(a) Run CubTop with the option -g to create a bitmap file of X.

(b) Run CubTop `properface` with the options -c -o -g to see what are the proper faces of X. Present the output files.

2.4 Create input files for each of the three elementary cubes displayed in the Figure 2.3. The list of elementary cubes in each file will contain one element only. Then perform the following functions.

(a) Repeat the step (b) of the previous exercise for each of your files.
(b) Run CubTop `union` with the options -c -o -g for the two elementary cubes in $\mathbf{R}^2$ to create their union Y.
(c) Repeat the step (b) of the previous exercise for the file of Y to display the free faces of Y.

2.5 Two combinatorial graphs $(\mathcal{V}_1, \mathcal{E}_1)$ and $(\mathcal{V}_2, \mathcal{E}_2)$ are *equivalent* if there exists a bijection $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that, for any $u, v \in \mathcal{V}_1$, $(u, v) \in \mathcal{E}_1$ if and only if $(f(u), f(v)) \in \mathcal{E}_2$.

Prove that any combinatorial graph which is a tree can be represented as a 1-dimensional cubical set X in the sense that it is equivalent to the combinatorial graph $(\mathcal{K}_0(X), \mathcal{E}(X))$ where $\mathcal{E}(X)$ are pairs of vertices of edges in $\mathcal{K}_0(X)$.

2.6 Give an example of a combinatorial graph which cannot be represented by a 1-dimensional cubical set.

2.7 Let Q^d be the unitary cube in Exercise 2.1 and let X be its one dimensional skeleton, i.e. the union of all edges of Q^d.

(a) For $d = 2, 3, 4, 5, 6$ determine the number of vertices of Q^d and the number of edges of Q^d (note that X has the same vertices and edges as Q^d).
(b) For the same values of d prepare the input files and run the program Homology to compute the dimensions r_0 and r_1 of $H_0(X)$ and $H_1(X)$ respectively.
(c) Make a guess, based on your empirical experience, on how the number of vertices, the number of edges and the dimensions r_0, r_1 are related together. Use more values of d, if what you computed in (a) and (b) is not sufficient for making a good guess.

2.8 Let $X_i := \left\{x \in \mathbf{R}^d \mid \|x\|_i \leq r\right\}$. For which values of r and which norms (see Section 12.1), $i = 0$, 1, or 2 is X_i a cubical set.

2.9 What are the elementary cells of the unit cube in $\mathbf{R}^3$?

2.10 The file exR2d.txt in the folder CubTop/Examples contains a list of elementary cells. Run CubTop on the file exR2d.txt with the option -g to see how the union U of those cells look like. Run CubTop `closedhull` with appropriate options to see the cubical set which is the closure of U.

2.2 The Algebra of Cubical Sets

In this section we finally present the formal definitions that we use to pass from the topology of a cubical set to the algebra of homology.

2.2.1 Cubical Chains

In Chapter 1 we associated algebraic structures with graphs. Given the set of vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots e_m\}$ we considered the 0-chains of the form

$$c = \alpha_1 \widehat{v_1} + \alpha_2 \widehat{v_2} \ldots \alpha_n \widehat{v_n}$$

and 1-chains of the form

$$d = \beta_1 \widehat{e_1} + \beta_2 \widehat{e_2} \ldots \beta_m \widehat{e_m}$$

respectively. The hat notation was used to distinguish between a vertex and an edge viewed as geometric objects and those viewed as algebraic objects. We shall now introduce an analogous algebraic structure for cubical sets of arbitrary dimension.

In analogy with what was done in Chapter 1, to each elementary k-cube $Q \in \mathcal{K}_k^d$ we identify an algebraic object $\widehat{Q}$ called an *elementary k-chain* of $\mathbf{R}^d$. The set of all elementary k-chains of $\mathbf{R}^d$ is denoted by

$$\widehat{\mathcal{K}}_k^d := \left\{ \widehat{Q} \mid Q \in \mathcal{K}_k^d \right\}$$

and the set of all *elementary chains* of $\mathbf{R}^d$ is given by

$$\widehat{\mathcal{K}}^d := \bigcup_{k=0}^{\infty} \widehat{\mathcal{K}}_k^d.$$

Given any *finite* collection $\{\widehat{Q}_1, \widehat{Q}_2, \ldots, \widehat{Q}_m\} \subset \mathcal{K}_k^d$ of k-dimensional elementary chains we are allowed to consider sums of the form

$$c = \alpha_1 \widehat{Q_1} + \alpha_2 \widehat{Q_2} \ldots \alpha_m \widehat{Q_m},$$

where α_i are arbitrary integers. If all the $\alpha_i = 0$, then we let $c = 0$. These can be thought of as our k-chains, the set of which is denoted by C_k^d. The addition of k-chains is naturally defined by

$$\sum \alpha_i \widehat{Q_i} + \sum \beta_i \widehat{Q_i} := \sum (\alpha_i + \beta_i) \widehat{Q_i}.$$

Observe that given an arbitrary k-chain $c = \sum_{i=0}^m \alpha_i \widehat{Q}_i$, there is an inverse element $-c = \sum_{i=0}^m (-\alpha_i) \widehat{Q}_i$ with the property that $c + (-c) = 0$. Therefore, C_k^d is an abelian group and in fact it is a free abelian group with basis $\widehat{\mathcal{K}}_k^d$.

The idea behind the presentation we just gave of the derivation of the abelian group of k-chains should be intuitively clear: each elementary cube is used to generate a basis element which we called an elementary chain and thus a chain is just defined in terms of a finite sum of elementary chains. There is, however, another prescription for using a set to generate a free abelian group. This involves viewing the chains as functions from $\mathcal{K}_k^d$ to $\mathbf{Z}$. (see Definition 13.67).

In particular, for each $Q \in \mathcal{K}_k^d$ define $\widehat{Q} : \mathcal{K}_k^d \to \mathbf{Z}$

$$\widehat{Q}(P) = \begin{cases} 1 & \text{if } P = Q \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and in a slight abuse of notation let $0 : \mathcal{K}_k^d \to \mathbf{Z}$ be the zero function, i.e. $0(Q) = 0$ for all $Q \in \mathcal{K}_k^d$. Refining the definition presented above, $\widehat{Q}$ is the *elementary chain dual* to the elementary cube Q. Since, the elementary chains take values in the integers we are allowed to take finite sums of them.

Definition 2.16 The group C_k^d of *k-dimensional chains* of $\mathbf{R}^d$ (k-chains for short) is the free abelian group generated by the elementary chains of $\mathcal{K}_k^d$. Thus the elements of C_k^d are functions $c : \mathcal{K}_k^d \to \mathbf{Z}$ such that $c(Q) = 0$ for all but a finite number of $Q \in \mathcal{K}_k^d$. In particular $\widehat{\mathcal{K}}_k^d$ is the basis for C_k^d. Using the notation of Definition 13.67

$$C_k^d := \mathbf{Z}(\mathcal{K}_k^d)$$

If $c \in C_k^d$ *then* $\dim c := k$.

Obviously, since the elementary cubes are contained in $\mathbf{R}^d$, for $k < 0$ and $k > d$ the set $\mathcal{K}_k = \emptyset$ and the corresponding group of k-chains is $C_k^d = 0$.

Observe that since $\mathcal{K}_k^d$ is infinite, C_k^d is an infinitely generated free abelian group. In practice we are interested in the chains localized to cubical sets and we will soon give an appropriate definition. However, as long as the localization is irrelevant it is convenient not to bind ourselves to a particular cubical set.

Recall that given an elementary cube Q its dual elementary chain is $\widehat{Q}$. Similarly, given an elementary chain $\widehat{Q}$ we refer to Q as its dual elementary cube. This is justified by the following proposition.

Proposition 2.17 *The map* $\phi : \mathcal{K}_k^d \to \widehat{\mathcal{K}}_k^d$ *given by* $\phi(Q) = \widehat{Q}$ *is a bijection.*

Proof. Since $\widehat{\mathcal{K}}_k^d$ is defined to be the image of ϕ it is obvious that ϕ is surjective. To prove injectivity assume that $P, Q \in \mathcal{K}_k^d$ and $\widehat{P} = \widehat{Q}$. This implies that

$$1 = \widehat{P}(P) = \widehat{Q}(P)$$

and hence that $P = Q$. □

Observe that the inverse of ϕ allows us to pass from an algebraic object, the elementary chain $\widehat{Q}$, to a well defined topological set, Q. We would like to be able to do this for general chains.

Definition 2.18 Let $c \in C^d_k$. The *support* of the chain c is the cubical set

$$|c| := \bigcup \left\{ Q \in \mathcal{K}^d_k \mid c(Q) \neq 0 \right\}.$$

Support has several nice geometric features.

Proposition 2.19 *Support satisfies the following properties.*

(i) $|c| = \emptyset$ *if and only if* $c = 0$.
(ii) Let $\alpha \in \mathbf{Z}$ *and* $c \in C^d_k$, *then*

$$|\alpha c| = \begin{cases} \emptyset & \text{if } \alpha = 0, \\ |c| & \text{if } \alpha \neq 0. \end{cases}$$

(iii) If $Q \in \mathcal{K}$, *then* $|\widehat{Q}| = Q$.
(iv) If $c_1, c_2 \in C^d_k$ *then* $|c_1 + c_2| \subset |c_1| \cup |c_2|$.

Proof. (i) In the case of the zero chain for every $Q \in \mathcal{K}^d_k$ the value $0(Q) = 0$, therefore $|c| = \emptyset$. On the other hand, if $|c| = \emptyset$, then there is no Q such that $c(Q) \neq 0$, therefore $c = 0$.

(ii) This follows directly from the definition of support and (i).

(iii) This too follows directly from the definition of chains and support.

(iv) Let $x \in |c_1 + c_2|$. Then $x \in Q$ for some $Q \in \mathcal{K}^d_k$ such that $(c_1+c_2)(Q) = c_1(Q)+c_2(Q) \neq 0$. It follows that either $c_1(Q) \neq 0$ or $c_2(Q) \neq 0$, hence $x \in |c_1|$ or $x \in |c_2|$. □

Example 2.20 It is not true in general that $|c_1 + c_2| = |c_1| \cup |c_2|$. Consider any chain c such that $|c| \neq \emptyset$. Observe that

$$\emptyset = |c - c| \neq |c| \cup |c| = |c| \neq \emptyset.$$

Similarly, notice that support does not define a bijection between the set of chains and cubical sets. In particular, for any chain c,

$$|2c| = |c|.$$

Consider an arbitrary chain $c \in C^d_k$. As was indicated above the set of elementary chains form a basis for C^d_k, thus it would be nice to have a simple formula that describes c in terms of the elements of $\widehat{\mathcal{K}}^d_k$. This is the motivation for the following definition which is analogous to the dot product in a vector space.

Definition 2.21 Consider $c_1, c_2 \in C^d_k$ where $c_1 = \sum_{i=1}^m \alpha_i \widehat{Q}_i$ and $c_2 = \sum_{i=1}^m \beta_i \widehat{Q}_i$. The *scalar product* of the chains c_1 and c_2 is defined as

$$\langle c_1, c_2 \rangle := \sum_{i=1}^m \alpha_i \beta_i.$$

Proposition 2.22 *The scalar product defines a mapping*

$$\begin{aligned}\langle\cdot,\cdot\rangle : C_k^d \times C_k^d &\to \mathbf{Z}\\ (c_1, c_2) &\mapsto \langle c_1, c_2\rangle\end{aligned}$$

which is bilinear.

Proof. We need to show that $\langle \alpha c_1 + \beta c_2, c_3\rangle = \alpha\langle c_1, c_3\rangle + \beta\langle c_2, c_3\rangle$ and $\langle c_1, \alpha c_2 + \beta c_3\rangle = \alpha\langle c_1, c_2\rangle + \beta\langle c_1, c_3\rangle$. For $j = 1, 2, 3$, let

$$c_j = \sum_{i=1}^{m} \gamma_{j,i}\widehat{Q}_i.$$

Then,

$$\begin{aligned}\langle \alpha c_1 + \beta c_2, c_3\rangle &= \left\langle \alpha\sum_{i=1}^{m}\gamma_{1,i}\widehat{Q}_i \;+\; \beta\sum_{i=1}^{m}\gamma_{2,i}\widehat{Q}_i, \sum_{i=1}^{m}\gamma_{3,i}\widehat{Q}_i\right\rangle\\ &= \left\langle \sum_{i=1}^{m}\left(\alpha\gamma_{1,i} + \beta\gamma_{2,i}\right)\widehat{Q}_i, \sum_{i=1}^{m}\gamma_{3,i}\widehat{Q}_i\right\rangle\\ &= \sum_{i=1}^{m}\left(\alpha\gamma_{1,i} + \beta\gamma_{2,i}\right)\gamma_{3,i}\\ &= \alpha\sum_{i=1}^{m}\gamma_{1,i}\gamma_{3,i} + \beta\sum_{i=1}^{m}\gamma_{2,i}\gamma_{3,i}\\ &= \alpha\left\langle \sum_{i=1}^{m}\gamma_{1,i}\widehat{Q}_i, \sum_{i=1}^{m}\gamma_{3,i}\widehat{Q}_i\right\rangle + \beta\left\langle \sum_{i=1}^{m}\gamma_{2,i}\widehat{Q}_i, \sum_{i=1}^{m}\gamma_{3,i}\widehat{Q}_i\right\rangle\\ &= \alpha\langle c_1, c_3\rangle + \beta\langle c_2, c_3\rangle.\end{aligned}$$

The proof of the other equality is nearly the same. □

The convenience of the scalar product notation comes from the fact that if $c = \sum_{i=1}^{m}\alpha_i\widehat{Q}_i$ then $\alpha_i = \langle c, \widehat{Q_i}\rangle$ (see exercise 2.11). Thus, the scalar product can be used to describe k-chains in terms of the canonical basis that consists of the elementary k-chains. This in turn allows us to relate the chain to topological data. In particular, Proposition 2.19 (ii) allows us to conclude that

$$|c| = \bigcup_{\langle c, \widehat{Q}_i\rangle \neq 0} Q_i. \tag{2.3}$$

While (2.3) is a nice formula, it does have its restrictions. The most obvious is that the support of the k-chain is only given in terms of k-dimensional cubes. On the other hand we know that cubes can be decomposed into lower dimensional faces. Furthermore, the construction of the boundary operator in Chapter 1 was based on this decomposition. As will be made precise in the next section, the cubical boundary operator takes k-chains to $k-1$-chains and for this reason we want some method to express the support of a k-chain in terms of lower dimensional cubes. This is the motivation for defining the following product.

Recall from Proposition 2.6 that the product of two elementary cubes is again an elementary cube. This can be used to construct a product on the chain level.

Definition 2.23 Given two elementary cubes $P \in \mathcal{K}_k^d$ and $Q \in \mathcal{K}_{k'}^{d'}$ set

$$\widehat{P} \diamond \widehat{Q} := \widehat{P \times Q}.$$

This definition extends to arbitrary chains $c_1 \in C_k^d$ and $c_2 \in C_{k'}^{d'}$ by

$$c_1 \diamond c_2 := \sum_{P \in \mathcal{K}_k, Q \in \mathcal{K}_{k'}} \langle c_1, \widehat{P} \rangle \langle c_2, \widehat{Q} \rangle \widehat{P \times Q}.$$

The chain $c_1 \diamond c_2 \in C_{k+k'}^{d+d'}$ is called the *cubical product* of c_1 and c_2.

Example 2.24 Let

$$P_1 = [0] \times [0,1],\ P_2 = [1] \times [0,1],\ P_3 = [0,1] \times [0],\ P_4 = [0,1] \times [1]$$

then $\widehat{P}_i \in \widehat{\mathcal{K}}_1^2$. Let $Q_1 = [-1,0]$ and $Q_2 = [0,1]$, then $\widehat{Q}_i \in \widehat{\mathcal{K}}_1^1$. This gives rise to chains $c_1 = \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_3 + \widehat{P}_4$ and $c_2 = \widehat{Q}_1 + \widehat{Q}_2$. By definition we have

$$\begin{aligned} c_1 \diamond c_2 = {} & \widehat{P_1 \times Q_1} + \widehat{P_2 \times Q_1} + \widehat{P_3 \times Q_1} + \widehat{P_4 \times Q_1} + \\ & \widehat{P_1 \times Q_2} + \widehat{P_2 \times Q_2} + \widehat{P_3 \times Q_2} + \widehat{P_4 \times Q_2} \end{aligned}$$

while

$$\begin{aligned} c_2 \diamond c_1 = {} & \widehat{Q_1 \times P_1} + \widehat{Q_1 \times P_2} + \widehat{Q_1 \times P_3} + \widehat{Q_1 \times P_4} + \\ & \widehat{Q_2 \times P_1} + \widehat{Q_2 \times P_2} + \widehat{Q_2 \times P_3} + \widehat{Q_2 \times P_4} \end{aligned}$$

Figure 2.4 indicates the support of the chains c_1, c_2, $c_1 \diamond c_2$ and $c_2 \diamond c_1$.

The cubical product has the following properties.

Proposition 2.25 *Let c_1, c_2, c_3 be any chains.*

(i) $c_1 \diamond 0 = 0 \diamond c_1 = 0$
(ii) $c_1 \diamond (c_2 + c_3) = c_1 \diamond c_2 + c_1 \diamond c_3$*, provided* $c_2, c_3 \in C_k^d$.
(iii) $(c_1 \diamond c_2) \diamond c_3 = c_1 \diamond (c_2 \diamond c_3)$
(iv) If $c_1 \diamond c_2 = 0$*, then* $c_1 = 0$ *or* $c_2 = 0$.
(v) $|c_1 \diamond c_2| = |c_1| \times |c_2|$

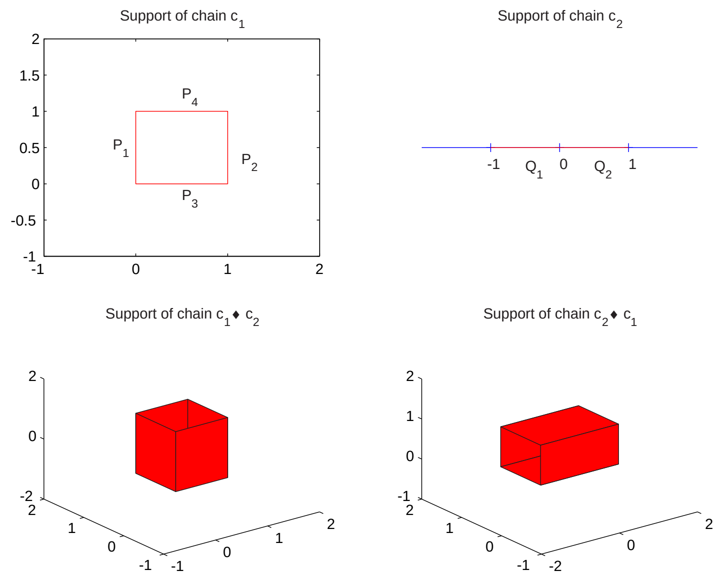


Fig. 2.4. The support of the chains c_1, c_2, $c_1 \diamond c_2$ and $c_2 \diamond c_1$.

Proof. (i) and (ii) follow immediately from the definition.

(iii) The proof is straightforward.

(iv) Assume that $c_1 = \sum_{i=1}^m \alpha_i \widehat{P}_i$ and $c_2 = \sum_{j=1}^n \beta_j \widehat{Q}_j$. Then

$$\sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j \widehat{P}_i \diamond \widehat{Q}_j = 0,$$

i.e. $\alpha_i \beta_j = 0$ for any $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$. It follows that

$$\Big(\sum_{i=1}^m \alpha_i^2\Big)\Big(\sum_{j=1}^n \beta_j^2\Big) = \sum_{i=1}^m \sum_{j=1}^n (\alpha_i \beta_j)^2 = 0,$$

hence $\sum_{i=1}^m \alpha_i^2 = 0$ or $\sum_{j=1}^n \beta_j^2 = 0$. Consequently $c_1 = 0$ or $c_2 = 0$.

(v) See exercise 2.12. □

Proposition 2.26 *Let $\widehat{Q}$ be an elementary cubical chain of $\mathbf{R}^d$ with $d > 1$. Then, there exist unique elementary cubical chains $\widehat{I}$ and $\widehat{P}$ with* $\operatorname{emb} I = 1$ *and* $\operatorname{emb} P = d - 1$ *such that*

$$\widehat{Q} = \widehat{I} \diamond \widehat{P}.$$

Proof. Since $\widehat{Q}$ is an elementary cubical chain, Q is an elementary cube, i.e.

$$Q = I_1 \times I_2 \times \cdots \times I_d.$$

Set $I := I_1$ and $P := I_2 \times I_3 \times \cdots \times I_d$, then $\widehat{Q} = \widehat{I} \diamond \widehat{P}$.

We still need to prove that this is the unique decomposition. If $\widehat{Q} = \widehat{J} \diamond \widehat{P'}$ for some $J \in \mathcal{K}^1$ and $P' \in \mathcal{K}^{d-1}$ then $\widehat{I_1 \times P} = \widehat{J \times P'}$ and from Proposition 2.17 we obtain $I_1 \times P = J \times P'$. Since $I_1, J \subset \mathbf{R}$, it follows that $I_1 = J$ and $P = P'$. $\square$

2.2.2 Cubical Chains in a Cubical Set

Having discussed chains in general, we move now to studying them in the context of a cubical set.

Definition 2.27 Let $X \subset \mathbf{R}^d$ be a cubical set. Let $\widehat{\mathcal{K}}_k(X) := \{\widehat{Q} \mid Q \in \mathcal{K}_k(X)\}$. $C_k(X)$ is the subgroup of C_k^d generated by the elements of $\widehat{\mathcal{K}}_k(X)$ and is referred to as the set of *k-chains of X*.

The reader can easily check that

$$C_k(X) = \left\{c \in C_k^d \mid |c| \subset X\right\}. \tag{2.4}$$

Since we know that $X \subset \mathbf{R}^d$, it is not necessary to write a superscript d in $\widehat{\mathcal{K}}_k(X)$ and $C_k(X)$.

It is left as an exercise that $\widehat{\mathcal{K}}_k(X)$ is a basis of $C_k(X)$, therefore it is a free abelian group. Moreover, since for any cubical set X the family $\mathcal{K}_k(X)$ is finite, $C_k(X)$ is finite dimensional. Finally, given any $c \in C_k(X)$ we have the decomposition

$$c = \sum_{Q_i \in \mathcal{K}_k(X)} \alpha_i \widehat{Q}_i,$$

where $\alpha_i := c(Q_i)$. This justifies our first informal definition of k-chains.

Utilizing the scalar product we have the following formula.

Proposition 2.28 *For any $c \in C_k(X)$*

$$c = \sum_{Q \in \mathcal{K}_k(X)} \langle c, \widehat{Q} \rangle \widehat{Q}$$

Remark 2.29 While the notation we are using for chains is consistent, some care must be taken when discussing 0-chains that are generated by elementary cubes in $\mathbf{R}$. Let $X \subset \mathbf{R}$ be a cubical set. Consider $\widehat{[1]} \in C_0(X)$. By definition it is the function

$$\widehat{[1]}(Q) = \begin{cases} 1 & \text{if } Q = [1] \\ 0 & \text{otherwise.} \end{cases}$$

while

$$2\widehat{[1]}(Q) = \begin{cases} 2 & \text{if } Q = [1] \\ 0 & \text{otherwise.} \end{cases}$$

This is different from $\widehat{[2]} \in C_0(X)$, since

$$\widehat{[2]}(Q) = \begin{cases} 1 & \text{if } Q = [2] \\ 0 & \text{otherwise.} \end{cases}$$

In particular $|\widehat{[1]}| = |2\widehat{[1]}| = \{1\} \subset \mathbf{R}$ while $|\widehat{[2]}| = \{2\} \subset \mathbf{R}$.

Finally, $0 \in C_k(X)$, the identity element of the group satisfies $|0| = \emptyset$, while $\widehat{[0]}$ is the dual of the vertex located at the origin, i.e. $|\widehat{0}| = \{0\} \subset \mathbf{R}$.

Example 2.30 Let $c = \widehat{A}_2 + \widehat{B}_1 - \widehat{B}_2 - \widehat{A}_1$, where

$$A_1 = [0] \times [0,1],\ B_1 = [1] \times [0,1],\ A_2 = [0,1] \times [0],\ B_2 = [0,1] \times [1]$$

Then $|c|$ is the contour of the square $Q := [0,1]^2$ shown in Figure 2.5. In addition we have chosen to give a geometric interpretation of the signs appearing in the expression for c. The orientation of a 1-dimensional elementary chain $\widehat{Q}$ is defined by the arrow from the lower end to the upper end of the unique nondegerate interval in it. The orientation of $-\widehat{Q}$ is the reverse. In Figure 2.5 we included this orientation indicated by the arrows. Thus, positive or negative elementary chains represent the direction in which an edge is traversed. For example, we think of $\widehat{A}_1$ as indicating moving along the edge from $(0,0)$ to $(0,1)$ while $-\widehat{A}_1$ suggests covering the edge in the opposite direction. With this in mind, c represents a counter-clockwise closed path around the square.

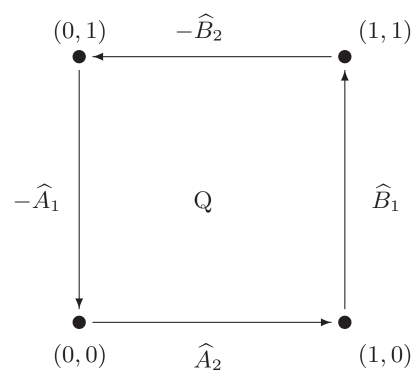


Fig. 2.5. Boundary of the unit square.

Example 2.31 Using the notation of the previous example, consider the chain $2c$. It is clear that $|2c| = |c|$ so both chains represent the same geometric object. The chain $2c$ can be interpreted as a path winding twice around the square in the counter-clockwise direction. Similarly, the chain

$$(\widehat{A}_1 + \widehat{B}_2) + (\widehat{A}_2 + \widehat{B}_1)$$

could be interpreted as a "sum" of two different paths along the boundary of the square connecting $(0,0)$ to $(1,1)$.

2.2.3 The Boundary Operator

Our aim in this section is to generalize the definition of the boundary operator given in Section 1.2 and to prove its basic properties.

Definition 2.32 Given $k \in \mathbf{Z}$, the *cubical boundary operator*

$$\partial_k : C_k^d \to C_{k-1}^d$$

is defined by induction on the embedding number. In order for ∂ to be a linear map it must be the case that

$$\partial_k \, 0 := 0.$$

Let $\widehat{Q} \in \widehat{\mathcal{K}}_k^1$, then Q is an elementary interval and hence $Q = [l] \in \mathcal{K}_0^1$ or $Q = [l, l+1] \in \mathcal{K}_1^1$ for some $l \in \mathbf{Z}$. Define

$$\partial_k \widehat{Q} := \begin{cases} 0 & \text{if } Q = [l], \\ \widehat{[l+1]} - \widehat{[l]} & \text{if } Q = [l, l+1]. \end{cases}$$

Now assume that $\widehat{Q} \in \widehat{\mathcal{K}}_k^d$ where $d > 1$. By Proposition 2.26 there exist unique elementary cubical chains $\widehat{I}, \widehat{P}$ with $\operatorname{emb} I = 1$ and $\operatorname{emb} P = d-1$ such that

$$\widehat{Q} = \widehat{I} \diamond \widehat{P}.$$

Define

$$\partial_k \widehat{Q} := \partial_{k_1} \widehat{I} \diamond \widehat{P} + (-1)^{\dim I} \widehat{I} \diamond \partial_{k_2} \widehat{P}. \tag{2.5}$$

Observe that $k_1 = \dim(I)$ and $k_2 = \dim(P)$. Finally, we extend the definition to all chains by linearity, i.e. if $c = \alpha_1 \widehat{Q}_1 + \alpha_2 \widehat{Q}_2 + \cdots + \alpha_m \widehat{Q}_m$ then

$$\partial_k c := \alpha_1 \partial_k \widehat{Q}_1 + \alpha_2 \partial_k \widehat{Q}_2 + \cdots + \alpha_m \partial_k \widehat{Q}_m.$$

Example 2.33 Let $Q = [l] \times [k]$. Then,

$$\begin{aligned} \partial_0 \widehat{Q} &= \partial_0 \widehat{[l]} \diamond \widehat{[k]} + (-1)^{\dim \widehat{[l]}} \widehat{[l]} \diamond \partial_0 \widehat{[k]} \\ &= 0 \diamond \widehat{[k]} + \widehat{[l]} \diamond 0 \\ &= 0 + 0. \end{aligned}$$

Example 2.34 Let $Q = [l, l+1] \times [k, k+1]$ (see Figure 2.6). Then,

$$\begin{aligned} \partial_2 \widehat{Q} &= \partial_1 \widehat{[l, l+1]} \diamond \widehat{[k, k+1]} + (-1)^{\dim \widehat{[l,l+1]}} \widehat{[l, l+1]} \diamond \partial_1 \widehat{[k, k+1]} \\ &= (\widehat{[l+1]} - \widehat{[l]}) \diamond \widehat{[k, k+1]} - \widehat{[l, l+1]} \diamond (\widehat{[k+1]} - \widehat{[k]}) \\ &= \widehat{[l+1]} \diamond \widehat{[k, k+1]} - \widehat{[l]} \diamond \widehat{[k, k+1]} - \widehat{[l, l+1]} \diamond \widehat{[k+1]} + \widehat{[l, l+1]} \diamond \widehat{[k]} \\ &= \widehat{[l+1] \times [k, k+1]} - \widehat{[l] \times [k, k+1]} + \widehat{[l, l+1] \times [k]} - \widehat{[l, l+1] \times [k+1]} \\ &= \widehat{B}_1 - \widehat{A}_1 + \widehat{A}_2 - \widehat{B}_2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= [l] \times [k, k+1] \\ B_1 &= [l+1] \times [k, k+1] \\ A_2 &= [l, l+1] \times [k] \\ B_2 &= [l, l+1] \times [k+1]. \end{aligned}$$

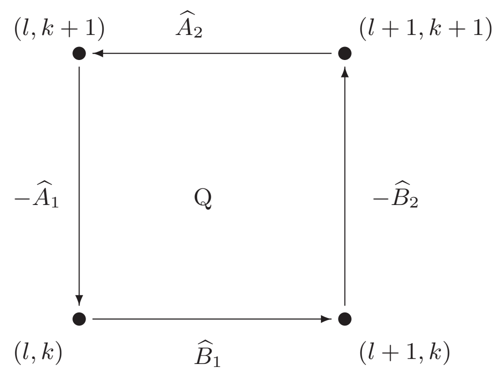


Fig. 2.6. Boundary of $[l, l+1] \times [k, k+1]$.

By definition the domain of ∂_k consists of the k-chains. Thus, if we know that $c \in C_k^d$, then it is redundant to write $\partial_k(c)$. What is worse, it is also inconvenient. Consider the defining equation (2.5) where we were forced to include the subscripts k_1 and k_2 which obviously are determined by the dimensions of the appropriate elementary cubical chains. Therefore, to simplify the presentation we shall simplify the notation ∂_k to ∂. The following proposition demonstrates the wisdom of this (try writing the proposition using subscripts for the boundary operator).

Proposition 2.35 *Let c and c' be cubical chains, then*

$$\partial(c \diamond c') = \partial c \diamond c' + (-1)^{\dim c} c \diamond \partial c'. \tag{2.6}$$

Proof. Let us begin by showing that it is sufficient to prove the result for elementary cubical chains. More precisely assume that for any $Q, Q' \in \mathcal{K}$,

$$\partial(\widehat{Q} \diamond \widehat{Q'}) = \partial\widehat{Q} \diamond \widehat{Q'} + (-1)^{\dim Q} \widehat{Q} \diamond \partial\widehat{Q'}. \tag{2.7}$$

Let $c = \sum_{i=1}^{m} \alpha_i \widehat{Q_i}$ and let $c' = \sum_{j=1}^{m'} \alpha'_j \widehat{Q'_j}$. Observe that $\dim c = \dim Q_i$. Now,

$$
\begin{aligned}
\partial(c \diamond c') &= \partial \left(\sum_{i=1}^{m} \alpha_i \widehat{Q_i} \diamond \sum_{j=1}^{m'} \alpha'_j \widehat{Q'_j} \right) \\
&\overset{1}{=} \partial \left(\sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \alpha'_j \widehat{Q_i} \diamond \widehat{Q'_j} \right) \\
&\overset{2}{=} \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \alpha'_j \partial \left(\widehat{Q_i} \diamond \widehat{Q'_j} \right) \\
&\overset{3}{=} \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \alpha'_j \left(\partial \widehat{Q_i} \diamond \widehat{Q'_j} + (-1)^{\dim Q_i} \widehat{Q_i} \diamond \partial \widehat{Q'_j} \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \alpha'_j \left(\partial \widehat{Q_i} \diamond \widehat{Q'_j} + (-1)^{\dim c} \widehat{Q_i} \diamond \partial \widehat{Q'_j} \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \partial \widehat{Q_i} \diamond \alpha'_j \widehat{Q'_j} + (-1)^{\dim c} \sum_{i=1}^{m} \sum_{j=1}^{m'} \alpha_i \alpha'_j \widehat{Q_i} \diamond \partial \widehat{Q'_j} \\
&= \partial c \diamond c' + (-1)^{\dim c} c \diamond \partial c'
\end{aligned}
$$

Equality 1 follows from the bilinearity of the cubical product. Equality 2 is due to the linearity of the boundary operator. Equality 3 follows from the assumption that (2.7) is satisfied.

Thus, to prove the proposition it is sufficient to verify (2.7). The proof will be done by induction on $d := \operatorname{emb} Q$.

If $d = 1$, then the result follows immediately from Definition 2.32.

If $d > 1$, then we can decompose Q as in Proposition 2.26, i.e. $Q = I \times P$ where $\operatorname{emb} I = 1$ and $\operatorname{emb} P = d - 1$. Then, by the definition of the boundary operator

$$
\begin{aligned}
\partial(\widehat{Q} \diamond \widehat{Q'}) &= \partial(\widehat{I} \diamond \widehat{P} \diamond \widehat{Q'}) \\
&= \partial \widehat{I} \diamond \widehat{P} \diamond \widehat{Q'} + (-1)^{\dim I} \widehat{I} \diamond \partial(\widehat{P} \diamond \widehat{Q'})
\end{aligned}
$$

Since $\widehat{P} \diamond \widehat{Q'}$ satisfies the induction assumption, we see that

$$
\begin{aligned}
\partial(\widehat{Q} \diamond \widehat{Q'}) &= \partial \widehat{I} \diamond \widehat{P} \diamond \widehat{Q'} + (-1)^{\dim I} \widehat{I} \diamond \left(\partial \widehat{P} \diamond \widehat{Q'} + (-1)^{\dim P} \widehat{P} \diamond \partial \widehat{Q'} \right) \\
&= \partial \widehat{I} \diamond \widehat{P} \diamond \widehat{Q'} + (-1)^{\dim I} \widehat{I} \diamond \partial \widehat{P} \diamond \widehat{Q'} + (-1)^{\dim I + \dim P} \widehat{I} \diamond \widehat{P} \diamond \partial \widehat{Q'} \\
&= \left(\partial \widehat{I} \diamond \widehat{P} + (-1)^{\dim I} \widehat{I} \diamond \partial \widehat{P} \right) \diamond \widehat{Q'} + (-1)^{\dim Q} \widehat{Q} \diamond \partial \widehat{Q'} \\
&= \partial \widehat{Q} \diamond \widehat{Q'} + (-1)^{\dim Q} \widehat{Q} \diamond \partial \widehat{Q'},
\end{aligned}
$$

where the last equality follows again from the definition of the boundary operator. □

By a straightforward induction argument we obtain the following corollary.

Corollary 2.36 *If $Q_1, Q_2, \dots, Q_m$ are elementary cubes, then*

$$\partial(\widehat{Q}_1 \diamond \widehat{Q}_2 \diamond \cdots \diamond \widehat{Q}_m) = \sum_{j=1}^{m} (-1)^{\sum_{i=1}^{j-1} \dim Q_i} \widehat{Q}_1 \diamond \cdots \diamond \widehat{Q}_{j-1} \diamond \partial \widehat{Q}_j \diamond \widehat{Q}_{j+1} \diamond \cdots \diamond \widehat{Q}_m.$$

From this corollary one can immediately obtain the following proposition.

Proposition 2.37 *Let $Q \in \mathbf{R}^d$ be an n-dimensional elementary cube with decomposition into elementary intervals given by $Q = I_1 \times I_2 \times \cdots I_d \in \mathbf{R}^d$ and let the 1-dimensional intervals in this decomposition be $I_{i_1}, I_{i_2}, \dots I_{i_n}$ with $I_{i_j} = [k_j, k_j + 1]$. For $j = 1, 2, \dots n$ let*

$$\begin{aligned} Q_j^- &:= I_1 \times \cdots \times I_{i_j - 1} \times [k_j] \times I_{i_j + 1} \times \cdots \times I_d \\ Q_j^+ &:= I_1 \times \cdots \times I_{i_j - 1} \times [k_j + 1] \times I_{i_j + 1} \times \cdots \times I_d \end{aligned}$$

denote the primary faces of Q. Then

$$\partial \widehat{Q} = \sum_{j=1}^{n} (-1)^{j-1} \left(\widehat{Q}_j^+ - \widehat{Q}_j^- \right).$$

The following proposition demonstrates one of the most important properties of the boundary operator.

Proposition 2.38

$$\partial \circ \partial = 0$$

Proof. Because ∂ is a linear operator it is enough to verify this property for elementary cubical chains. Again, the proof is by induction on the embedding number.

Let Q be an elementary interval. If $Q = [l]$, then by definition $\partial \widehat{Q} = 0$ so $\partial(\partial \widehat{Q}) = 0$. If $Q = [l, l+1]$, then

$$\begin{aligned} \partial(\partial \widehat{Q}) &= \partial(\partial \widehat{[l, l+1]}) \\ &= \partial(\widehat{[l+1]} - \widehat{[l]}) \\ &= \partial \widehat{[l+1]} - \partial \widehat{[l]} \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

Now assume that $Q \in \mathcal{K}^d$ for $d > 1$. Then by Proposition 2.26 we can write $Q = I \times P$ where $\operatorname{emb} I = 1$ and $\operatorname{emb} P = d - 1$. So

$$\begin{aligned}
\partial(\partial\widehat{Q}) &= \partial(\partial(\widehat{I \times P})) \\
&= \partial(\partial(\widehat{I} \diamond \widehat{P})) \\
&= \partial\left(\partial\widehat{I} \diamond \widehat{P} + (-1)^{\dim \widehat{I}}\widehat{I} \diamond \partial\widehat{P}\right) \\
&= \partial\left(\partial\widehat{I} \diamond \widehat{P}\right) + (-1)^{\dim \widehat{I}}\partial\left(\widehat{I} \diamond \partial\widehat{P}\right) \\
&= \partial\partial\widehat{I} \diamond \widehat{P} + (-1)^{\dim \partial\widehat{I}}\partial\widehat{I} \diamond \partial\widehat{P} + (-1)^{\dim \widehat{I}}\left(\partial\widehat{I} \diamond \partial\widehat{P} + (-1)^{\dim \widehat{I}}\widehat{I} \diamond \partial\partial\widehat{P}\right) \\
&= (-1)^{\dim \partial\widehat{I}}\partial\widehat{I} \diamond \partial\widehat{P} + (-1)^{\dim \widehat{I}}\partial\widehat{I} \diamond \partial\widehat{P}.
\end{aligned}$$

The last step uses the induction hypothesis that the proposition is true if the embedding number is less than d.

Observe that if $\dim \widehat{I} = 0$, then $\partial\widehat{I} = 0$ in which case we have that each term in the sum is 0 and hence $\partial\partial\widehat{Q} = 0$. On the other hand, if $\dim \widehat{I} = 1$, then $\dim \partial\widehat{I} = 0$ and hence the two terms cancel each other giving the desired result. □

Example 2.34 and Figure 2.6 might suggest that the algebraic and topological boundaries are closely related. We would like to know whether or not

$$|\partial c| = \operatorname{bd} |c|?$$

Exercise 2.15 shows that this is false. What is true in general is the following proposition.

Proposition 2.39 *For any chain $c \in C_k^d$*

$$|\partial c| \subset |c|.$$

Moreover, $|\partial c|$ is contained in the $(k-1)$-dimensional skeleton *of $|c|$, i.e. the union of $(k-1)$-dimensional faces of $|c|$.*

Proof. First consider the case when $c = \widehat{Q}$, where $Q \in \mathcal{K}_k$. It follows from Exercise 2.18 that $|\partial\widehat{Q}| \subset \bigcup \mathcal{K}_{k-1}(Q) \subset Q = |\widehat{Q}|$. If c is arbitrary, then $c = \sum_i \alpha_i \widehat{Q}_i$ for some $\alpha_i \neq 0$ and

$$|\partial c| = |\sum_i \alpha_i \partial\widehat{Q}_i| \subset \bigcup_i |\partial\widehat{Q}_i| \subset \bigcup_i |\widehat{Q}_i| = |c|.$$

□

We now go back to cubical chains in the setting of a fixed cubical set X. The first observation is that the boundary operator maps chains in X into chains in X. More precisely we have the following proposition.

Proposition 2.40 *Let $X \subset \mathbf{R}^d$ be a cubical set, then*

$$\partial_k(C_k(X)) \subset C_{k-1}(X).$$

Proof. Let $c \in C_k(X)$. Then by Formula (2.4) $|c| \subset X$ and by Proposition 2.39 $|\partial_k(c)| \subset |c| \subset X$. Therefore $\partial_k(c) \in C_{k-1}(X)$. □

An immediate consequence of this Proposition is that the restriction of the operator ∂ to chains in X, $\partial_k^X : C_k(X) \to C_{k-1}(X)$, given by

$$\partial_k^X(c) := \partial_k(c)$$

makes sense. This justifies the following definition

Definition 2.41 The boundary operator for the cubical set X is defined to be

$$\partial_k^X : C_k(X) \to C_{k-1}(X)$$

obtained by restricting $\partial_k : C_k^d \to C_{k-1}^d$ to $C_k(X)$.

In the sequel we will frequently omit the superscript X in ∂_k^X whenever X is clear from the context.

Definition 2.42 The *cubical chain complex* for the cubical set $X \subset \mathbf{R}^d$ is

$$\mathcal{C}(X) := \{C_k(X), \partial_k^X\}_{k \in \mathbf{Z}},$$

where $C_k(X)$ are the groups of cubical k-chains generated by $\mathcal{K}_k(X)$ and ∂_k^X is the cubical boundary operator restricted to X.

2.2.4 Homology of Cubical Sets

We are now ready to give the most important definitions of this book. Let $X \subset \mathbf{R}^d$ be a cubical set. A k-chain $z \in C_k(X)$ is called a *cycle* in X if $\partial z = 0$. Recall from Definition 13.36 that the kernel of a linear map is the set of elements which are sent to zero and is a subgroup of the domain. Thus the set of all k-cycles in X, which is denoted by $Z_k(X)$, is $\ker \partial_k^X$ and forms a subgroup of $C_k(X)$. Because of its importance we explicitly summarize these comments via the following set of relations

$$Z_k(X) := \ker \partial_k^X = C_k(X) \cap \ker \partial_k \subset C_k(X). \tag{2.8}$$

A k-chain $z \in C_k(X)$ is called a *boundary* in X if there exists $c \in C_{k+1}(X)$ such that $\partial c = z$. Thus, the set of boundary elements in $C_k(X)$, which is denoted by $B_k(X)$, consists of the image of ∂_{k+1}^X. Since ∂_{k+1}^X is a homomorphism, $B_k(X)$ is a subgroup of $C_k(X)$. Again, these comments can be summarized by

$$B_k(X) := \operatorname{im} \partial_{k+1}^X = \partial_{k+1}(C_{k+1}(X)) \subset C_k(X). \tag{2.9}$$

By Proposition 2.38, $\partial c = z$ implies $\partial z = \partial^2 c = 0$. Hence every boundary is a cycle and thus $B_k(X)$ is a subgroup of $Z_k(X)$. We are interested in cycles which are not boundaries. We want to treat cycles which are boundaries as trivial. In order to give the non-trivial cycles an algebraic structure we introduce an equivalence relation. We say that two cycles $z_1,\ z_2 \in Z_k(X)$ are *homologous* and we write $z_1 \sim z_2$ if $z_1 - z_2$ is a boundary in X, i.e. $z_1 - z_2 \in B_k(X)$. The equivalence classes are elements of the quotient group $Z_k(X)/B_k(X)$ (see Chapter 13 for the definition of the quotient group).

Definition 2.43 The k-th *cubical homology group* or briefly the k-th homology group of X is the quotient group

$$H_k(X) := Z_k(X)/B_k(X).$$

The homology of X is the collection of all homology groups of X. The shorthand notation for this is

$$H_*(X) := \{H_k(X)\}_{k\in\mathbf{Z}}.$$

As was emphasized in the introduction, we will use the homology groups of the cubical set X to gain information about the topological structure of X. Thus, the elements of $H_*(X)$ play an important role. However, we used chains to pass from the topology of X to the homology $H_*(X)$ and therefore, we often find ourselves discussing the elements of $H_*(X)$ in terms of representative chains. For this reason we introduce the following notation.

Definition 2.44 Given $z \in C_k(X)$, $[z]_X \in H_k(X)$ is the homology class of z in X. To simplify the notation, if the cubical set X is clear from the context of the discussion then we shall let $[z] := [z]_X$.

Example 2.45 Let $X = \emptyset$. Then $C_k(X) = 0$ for all k and hence

$$H_k(X) = 0 \quad k = 0, 1, 2, \ldots.$$

Example 2.46 Let $X = \{x_0\} \subset \mathbf{R}^d$ be a cubical set consisting of a single point. Then $x_0 = [l_1] \times [l_2] \times \cdots \times [l_d]$. Thus,

$$C_k(X) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $Z_0(X) \cong C_0(X) = \mathbf{Z}$. Since $C_1 = 0$, $B_0 = 0$ and therefore, $H_0(X) \cong \mathbf{Z}$. Since, $C_k(X) = 0$ for all $k \geq 1$, $H_k(X) = 0$ for all $k \geq 1$. Therefore,

$$H_k(X) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.47 Recall the cubical set

$$\Gamma^1 = [0] \times [0,1] \cup [1] \times [0,1] \cup [0,1] \times [0] \cup [0,1] \times [1]$$

The sets of elementary cubes are

$$\mathcal{K}_0(\Gamma^1) = \{[0] \times [0], [0] \times [1], [1] \times [0], [1] \times [1]\}$$
$$\mathcal{K}_1(\Gamma^1) = \{[0] \times [0,1], [1] \times [0,1], [0,1] \times [0], [0,1] \times [1]\}$$

Thus, the bases for the sets of chains are

$$\begin{aligned}\widehat{\mathcal{K}}_0(\Gamma^1) &= \{\widehat{[0]\times[0]}, \widehat{[0]\times[1]}, \widehat{[1]\times[0]}, \widehat{[1]\times[1]}\}\\ &= \{\widehat{[0]}\diamond\widehat{[0]}, \widehat{[0]}\diamond\widehat{[1]}, \widehat{[1]}\diamond\widehat{[0]}, \widehat{[1]}\diamond\widehat{[1]}\}\\ \widehat{\mathcal{K}}_1(\Gamma^1) &= \{\widehat{[0]\times[0,1]}, \widehat{[1]\times[0,1]}, \widehat{[0,1]\times[0]}, \widehat{[0,1]\times[1]}\}\\ &= \{\widehat{[0]}\diamond\widehat{[0,1]}, \widehat{[1]}\diamond\widehat{[0,1]}, \widehat{[0,1]}\diamond\widehat{[0]}, \widehat{[0,1]}\diamond\widehat{[1]}\}\end{aligned}$$

To compute the boundary operator we need to compute the boundary of the basis elements.

$$\begin{aligned}\partial(\widehat{[0]}\diamond\widehat{[0,1]}) &= -\widehat{[0]}\diamond\widehat{[0]} + \widehat{[0]}\diamond\widehat{[1]}\\ \partial(\widehat{[1]}\diamond\widehat{[0,1]}) &= -\widehat{[1]}\diamond\widehat{[0]} + \widehat{[1]}\diamond\widehat{[1]}\\ \partial(\widehat{[0,1]}\diamond\widehat{[0]}) &= -\widehat{[0]}\diamond\widehat{[0]} + \widehat{[1]}\diamond\widehat{[0]}\\ \partial(\widehat{[0,1]}\diamond\widehat{[1]}) &= -\widehat{[0]}\diamond\widehat{[1]} + \widehat{[1]}\diamond\widehat{[1]}.\end{aligned}$$

We can put this into the form of a matrix

$$\partial_1 = \begin{bmatrix} -1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}.$$

To understand $Z_1(\Gamma^1)$ we need to know $\ker\partial_1$, i.e. we need to solve the equation

$$\begin{bmatrix} -1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

This in turn means solving

$$\begin{bmatrix}-\alpha_1-\alpha_3\\ \alpha_1-\alpha_4\\ -\alpha_2+\alpha_3\\ \alpha_2+\alpha_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

which gives

$$\alpha_1 = -\alpha_2 = -\alpha_3 = \alpha_4.$$

Hence

$$Z_1(\Gamma^1) = \left\{\alpha[1,-1,-1,1]^T \mid \alpha\in\mathbf{Z}\right\},$$

i.e. $Z_1(\Gamma^1)$ is generated by

$$\widehat{[0]}\diamond\widehat{[0,1]} - \widehat{[1]}\diamond\widehat{[0,1]} - \widehat{[0,1]}\diamond\widehat{[0]} + \widehat{[0,1]}\diamond\widehat{[1]}.$$

Since, $C_2(\Gamma^1) = 0$, $B_1(\Gamma^1) = 0$ and hence

$$H_1(\Gamma^1) = Z_1(\Gamma^1) \cong \mathbf{Z}.$$

We turn to computing $H_0(\Gamma^1)$. First observe that there is no solution to the equation

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies that $\widehat{[0]} \diamond \widehat{[0]} \notin B_0(\Gamma^1)$. On the other hand

$$\begin{aligned} \partial\left(\widehat{[0]} \diamond \widehat{[0,1]}\right) &= -\widehat{[0]} \diamond \widehat{[0]} + \widehat{[0]} \diamond \widehat{[1]} \\ \partial\left(\widehat{[0]} \diamond \widehat{[0,1]} + \widehat{[0,1]} \diamond \widehat{[1]}\right) &= -\widehat{[0]} \diamond \widehat{[0]} + \widehat{[1]} \diamond \widehat{[1]} \\ \partial\left(\widehat{[0]} \diamond \widehat{[0,1]} + \widehat{[0,1]} \diamond \widehat{[1]} - \widehat{[1]} \diamond \widehat{[0,1]}\right) &= -\widehat{[0]} \diamond \widehat{[0]} + \widehat{[1]} \diamond \widehat{[0]}. \end{aligned}$$

Thus,

$$\{\widehat{[0]} \diamond \widehat{[0]} - \widehat{[0]} \diamond \widehat{[1]}, \widehat{[0]} \diamond \widehat{[0]} - \widehat{[1]} \diamond \widehat{[0]}, \widehat{[0]} \diamond \widehat{[0]} - \widehat{[1]} \diamond \widehat{[1]}\} \subset B_0(\Gamma^1).$$

In particular, all the elementary chains are homologous, i.e. $\widehat{[0]} \diamond \widehat{[0]} \sim \widehat{[0]} \diamond \widehat{[1]} \sim \widehat{[1]} \diamond \widehat{[0]} \sim \widehat{[1]} \diamond \widehat{[1]}$.

Now consider an arbitrary chain $z \in C_0(\Gamma^1)$. Then,

$$z = \alpha_1 \widehat{[0]} \diamond \widehat{[0]} + \alpha_2 \widehat{[0]} \diamond \widehat{[1]} + \alpha_3 \widehat{[1]} \diamond \widehat{[0]} + \alpha_4 \widehat{[1]} \diamond \widehat{[1]}.$$

So on the level of homology

$$\begin{aligned} [z]_{\Gamma^1} &= \left[\alpha_1 \widehat{[0]} \diamond \widehat{[0]} + \alpha_2 \widehat{[0]} \diamond \widehat{[1]} + \alpha_3 \widehat{[1]} \diamond \widehat{[0]} + \alpha_4 \widehat{[1]} \diamond \widehat{[1]}\right]_{\Gamma^1} \\ &= \alpha_1 \left[\widehat{[0]} \diamond \widehat{[0]}\right]_{\Gamma^1} + \alpha_2 \left[\widehat{[0]} \diamond \widehat{[1]}\right]_{\Gamma^1} + \alpha_3 \left[\widehat{[1]} \diamond \widehat{[0]}\right]_{\Gamma^1} + \alpha_4 \left[\widehat{[1]} \diamond \widehat{[1]}\right]_{\Gamma^1} \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \left[\widehat{[0]} \diamond \widehat{[0]}\right]_{\Gamma^1} \end{aligned}$$

where the last equality comes from that fact that all the elementary chains are homologous. Therefore, we can think of every element of $H_0(\Gamma^1) = Z_0(\Gamma^1)/B_0(\Gamma^1)$ as being generated by $\widehat{[0]} \diamond \widehat{[0]}$ and thus $\dim H_0(\Gamma^1) = 1$.

In particular, we have proven that

$$H_k(\Gamma^1) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

We could continue in this fashion for a long time computing homology groups, but as the reader hopefully has already seen this is a rather time consuming process. Furthermore, even if one takes a simple set such as

$$X = [0,1] \times [0,1] \times [0,1] \times [0,1]$$

the number of elementary cubes is quite large and the direct computation of its homology is quite tedious. Thus, we need to develop more efficient methods.

Exercises

2.11 Let $c \in C_k^d$.

1. Show that for any $Q \in \mathcal{K}_k^d$,

$$c(Q) = \langle c, \widehat{Q} \rangle.$$

2. Justify the following equalities

$$c = \sum_{Q \in \mathcal{K}_k^d} c(Q)\widehat{Q} = \sum_{Q \in \mathcal{K}_k^d} \langle c, \widehat{Q} \rangle \widehat{Q}.$$

2.12 Prove Proposition 2.25(v).

2.13 For each Q determine the elementary cubes I and P of Proposition 2.26 that satisfy $\widehat{Q} = \widehat{I} \diamond \widehat{P}$. What are the dimensions of I and P?

(a) Let $Q = [2,3] \times [1,2] \times [0,1]$.
(b) Let $Q = [-1] \times [0,1] \times [2]$.

2.14 Let $c \in C_k^d$. Show that

$$|c| := \bigcup \left\{ Q \in \mathcal{K}^d \mid \langle c, \widehat{Q} \rangle \neq 0 \right\}.$$

2.15 Let

$$c_1 = \widehat{[-1,0]} \times \widehat{[0]} + \widehat{[0,1]} \times \widehat{[0]} + \widehat{[0]} \times \widehat{[-1,0]} + \widehat{[0]} \times \widehat{[0,1]}$$

and

$$c_2 = \widehat{[-1,0]} \times \widehat{[0]} + \widehat{[0,1]} \times \widehat{[0]} + \widehat{[0]} \times \widehat{[0,1]}.$$

Consider the sets $X = |c_1|$ and $Y = |c_2|$ in $\mathbf{R}^2$.

(a) Make a sketch of X and Y. Try to define the *topological boundary* of those sets by analogy to the discussion in Chapter 1. Note that this concept was not explicitly defined in that chapter and it was only used in very simple cases. According to your definition, is the vertex $(0,0)$ a part of the boundary of X or Y?
(b) Find ∂c_1 and ∂c_2. Determine the sets $|\partial c_1|$ and $|\partial c_2|$. Note that $(0,0) \in |\partial c_2|$ but $(0,0) \notin |\partial c_1|$. This shows that one has to be careful about the relation between the concept of algebraic boundary and topological boundary whatever this second one could mean.

2.16 Create the input files for the program CubTop of the sets X and Y given in the previous exercise.

(a) Call CubTop for each file with the functions `freefaces` and `topobound` and the option -g. View the bitmap files. Does any of them fit your answer to the previous question?
(b) Repeat the same exercise for the file exC2d.txt in the folder CubTop/Examples.

2.17 Let $Q = [0,1]^d \subset \mathbf{R}^d$ and put

$$A_j := \{x \in Q \mid x_j = 0\},$$
$$B_j := \{x \in Q \mid x_j = 1\}.$$

Show that

$$\partial \widehat{Q} = \sum_{j=1}^{d} (-1)^j (\widehat{A_j} - \widehat{B_j}) \ .$$

2.18 Let $Q = Q_1 \times Q_2 \times \cdots \times Q_m$ be an elementary cube of dimension k. Show that the support of each nonzero term of the sum in Corollary 2.36 is the union of two parallel $(k-1)$-dimensional faces of Q.

2.19 Let P be an elementary k-dimensional cube. Let Q be a $k-1$-dimensional face of P. Show that

$$\langle \partial \widehat{P}, \widehat{Q} \rangle = \pm 1.$$

2.20 Prove that if $\widehat{Q}_1, \widehat{Q}_2, \ldots, \widehat{Q}_m$ are elementary cubical chains, then

$$\partial(\widehat{Q}_1 \diamond \widehat{Q}_2 \diamond \cdots \diamond \widehat{Q}_m) = \sum_{j=1}^{m} (-1)^{\sum_{i=1}^{j-1} \dim Q_i} \widehat{Q}_1 \diamond \cdots \diamond \widehat{Q}_{j-1} \diamond \partial \widehat{Q}_j \diamond \widehat{Q}_{j+1} \diamond \cdots \diamond \widehat{Q}_m.$$

2.21 Let $X = \{0\} \times [-1,1] \cup [-1,1] \times \{0\} \subset \mathbf{R}^2$. Determine the cubical chain complex $\mathcal{C}(X)$ and compute $H_*(X)$.

2.22 Let X consist of the 1-dimensional faces of $[0,2] \times [0,1]$. Determine the cubical chain complex $\mathcal{C}(X)$ and compute $H_*(X)$.

2.23 Let Γ^2 be the cubical set consisting of the 2-dimensional faces of $[0,1]^2$. Determine the cubical chain complex $\mathcal{C}(\Gamma^2)$ and compute $H_*(\Gamma^2)$.

2.24 Let X be a cubical set obtained by removing the set $(1,2) \times (1,2) \times [0,1]$ from the solid rectangle $[0,3] \times [0,3] \times [0,1]$. Let T be the union of the free faces of X (compare this set with a torus discussed in Example 11.9).

(a) Prepare the data file for computing the chain complex $\mathcal{C}(X)$ of X by the Homology program . Run the program to find $\mathcal{C}(X)$ and $H_*(X, \mathbf{Z})$.

(b) Determine $\mathcal{C}(T)$ and compute $H_*(T)$.

2.25 Let L be the cubical set presented in Figure 1.2 (L is defined in the file maze.bmp). Run the Homology program to find the homology of L. Open two gates (i.e. remove two pixels) in opposite walls of the labyrinth and again run the program to find the homology of what is left. Make a guess about the solvability of the labyrinth. i.e. a possibility of passing inside from one gate to another without crossing a wall.

2.26 Let K be the cubical set defined in the file kleinbot.cub in the folder Examples of Homology program . Run the program to find the homology of K with respect to

(a) coefficients in Z;
(b) coefficients in $\mathbf{Z}_p$ for $p = 2, 3, 5, 7$.

Derive a conjecture about the result for any prime number p.

2.3 Connected Components and $H_0(X)$

This Chapter began with a discussion of cubical sets which form a very special class of topological spaces. We then moved on to the combinatorics and algebra associated with these spaces, and finally, defined the homology of a cubical set. Though there is much more to be said, we have defined the essential steps in moving from topology to homology. It is worth noting how little topology was involved in the process; in fact most of our discussion has revolved around the algebra induced by combinatorial data. The opposite sequence of relations has not been addressed; what do the homology groups imply about the topology of the set. In this section we will begin to move in this direction. We will show that the zero dimensional homology group measures the number of connected components of the cubical set. However to do so, we need to show first that in the case of cubical sets there is a purely combinatorial way to describe the topological concepts of connectedness and connected components.

Recall that for any topological space X and any point $x \in X$ the union of all connected subsets of X containing x is a connected subset of X (see Theorem 12.54). It is called the *connected component of x in X*. We will denote it by $\mathrm{cc_X(x)}$.

Theorem 2.48 *For any $x, y \in X$ either* $\mathrm{cc_X(x)} = \mathrm{cc_X(y)}$ *or* $\mathrm{cc_X(x)} \cap \mathrm{cc_X(y)} = \emptyset$.

Proof. Assume that $\mathrm{cc_X(x)} \cap \mathrm{cc_X(y)} \neq \emptyset$. Then by Theorem 12.54, $\mathrm{cc_X(x)} \cup \mathrm{cc_X(y)}$ is connected. Since it contains both x and y, it must be the case that $\mathrm{cc_X(x)} \cup \mathrm{cc_X(y)} \subset \mathrm{cc_X(x)}$ and $\mathrm{cc_X(x)} \cup \mathrm{cc_X(y)} \subset \mathrm{cc_X(y)}$. It follows that

$$\mathrm{cc_X(x)} = \mathrm{cc_X(x)} \cup \mathrm{cc_X(y)} = \mathrm{cc_X(y)}.$$

□

Let X be a cubical set. Let us observe first the following simple proposition.

Proposition 2.49 *For every $x \in X$ there exists a vertex $V \in \mathcal{K}_0(X)$ such that* $\mathrm{cc_X(x)} = \mathrm{cc_X(V)}$.

Proof. By Proposition 2.15(i) there exists an elementary cube Q such that $x \in \overset{\circ}{Q}$. Therefore $\overset{\circ}{Q} \cap X \neq \emptyset$, and it follows from Proposition 2.15(vi) that $Q \subset X$. Let V be any vertex of Q. Since Q as a cube is connected (see Theorem 12.55), $Q \subset \mathrm{cc_X(x)}$ and consequently $V \in \mathrm{cc_X(x)}$. Hence $\mathrm{cc_X(V)} \cap \mathrm{cc_X(x)} \neq \emptyset$ and by Theorem 2.48 $\mathrm{cc_X(V)} = \mathrm{cc_X(x)}$. □

Corollary 2.50 *A cubical set can have only a finite number of connected components.*

Proof. By Proposition 2.49 every connected component of a cubical set is a connected component of one of its vertices and a cubical set has only a finite number of vertices. □

Definition 2.51 A sequence of vertices $V_0, V_1, \ldots V_n \in \mathcal{K}_0(X)$ is an *edge path in X* if there exist edges $E_1, E_2, \ldots E_n \in \mathcal{K}_1(X)$ such that V_{i-1}, V_i are the two faces of E_i for $i = 1, 2, \ldots, n$. For $V, V' \in \mathcal{K}_0(X)$ we write $V \sim_X V'$ if there exists an edge path $V_0, V_1, \ldots V_n \in \mathcal{K}_0(X)$ in X such that $V = V_0$ and $V' = V_n$. We say that X is *edge connected* if $V \sim_X V'$ for any $V, V' \in \mathcal{K}_0(X)$.

As Exercise 2.27 indicates $\sim_X$ is an equivalence relation. Also left to the exercises is the proof of the following proposition.

Proposition 2.52 *1. Every elementary cube is edge connected.*
2. If X and Y are edge connected cubical sets and $X \cap Y \neq \emptyset$, then $X \cup Y$ is edge connected.

Proposition 2.53 *Assume that $V \sim_X V'$ for some $V, V' \in \mathcal{K}_0(X)$. Then there exists a chain $c \in C_1(X)$ such that $|c|$ is connected and $\partial c = \widehat{V}' - \widehat{V}$.*

Proof. Let $V_0, V_1, \ldots V_n \in \mathcal{K}_0(X)$ be an edge path from $V = V_0$ to $V' = V_n$ and let $E_1, E_2, \ldots E_n \in \mathcal{K}_1(X)$ be the corresponding edges. Without loss of generality we may assume that the edge path is minimal. Then any two edges as well as any two vertices in the path are different. We will show that for some coefficients $\alpha_i \in \{-1, 1\}$ the chain

$$c := \sum_{i=1}^{n} \alpha_i \widehat{E}_i$$

satisfies the conclusion of the proposition. We do so by induction in n. If $n = 1$, then $\partial E_1 = \pm(\widehat{V_1} - \widehat{V_0})$. Taking $c = \alpha_1 \widehat{E}_1$ with an appropriate coefficient $\alpha_1 \in \{-1, 1\}$ we get $\partial(c) = \widehat{V_1} - \widehat{V_0}$. Since $|c| = |\alpha_1 E_1| = E_1$, it is connected.

Consider in turn the second step of the induction argument. Let

$$c' := \sum_{i=1}^{n-1} \alpha_i \widehat{E}_i$$

with coefficients chosen so that $\partial c' = \widehat{V}_{n-1} - \widehat{V}_0$ and $|c'|$ is connected. Choose α_n so that $\partial(\alpha_n E_n) = \widehat{V_n} - \widehat{V_{n-1}}$. Then obviously $\partial c = \widehat{V}_n - \widehat{V}_0$. Since $|c| = |c'| \cup E_n$ and $|c'| \cap E_n \neq \emptyset$, it follows from Theorem 2.48 that $|c|$ is connected. □

For $x \in X$ we define the *edge connected component of* x *in* X as the union of all edge connected cubical subset of X which contain x. We denote it by $\mathrm{ecc_X(x)}$.

Since the number of cubical subsets of a given cubical set is finite, we may use an induction argument based on Proposition 2.52 to prove the following proposition.

Proposition 2.54 *For any* $x \in X$, $\mathrm{ecc_X(x)}$ *is edge connected.*

The same argument as in the case of connected components shows that we have the following counterpart of Theorem 2.48.

Proposition 2.55 *For any* $x, y \in X$ *either*

$$\mathrm{ecc_X(x)} = \mathrm{ecc_X(y)} \text{ or } \mathrm{ecc_X(x)} \cap \mathrm{ecc_X(y)} = \emptyset.$$

Theorem 2.56 *A cubical set* X *is connected if and only if it is edge connected.*

Proof. Assume first that X is edge connected. Let $V \in X$ be a vertex. It is enough to show that $\mathrm{cc_X(V) = X}$, because $\mathrm{cc_X(V)}$ is connected. Since $\mathrm{cc_X(V) \subset X}$, we need only to show the opposite inclusion. Thus let $x \in X$. By Proposition 2.49 we can select another vertex $W \in X$ such that $x \in \mathrm{cc_X(x) = cc_X(W)}$. By Proposition 2.53 there exists a chain $c \in C_1(X)$ such that $\partial c = \widehat{V} - \widehat{W}$ and $|c|$ is connected. Since $V, W \in |c|$, it follows that $\mathrm{cc_X(W) = cc_X(V)}$. Therefore $x \in \mathrm{cc_X(V)}$ what we needed to prove.

To prove the opposite implication assume that X is not edge connected. Then there exist vertices V_0, V_1 in X such that $\mathrm{ecc_X(V_0) \cap ecc_X(V_1)} = \emptyset$. Put $X_0 := \mathrm{ecc_X(V_0)}$ and

$$X_1 := \bigcup \{\mathrm{ecc_X(V) \mid V \in \mathcal{K}_0(X) \text{ and } ecc_X(V) \cap ecc_X(V_0)} = \emptyset\}.$$

The sets X_0, X_1 are disjoint, non-empty closed subsets of X. We will show that $X = X_0 \cup X_1$. Let $x \in X$. Let Q be an elementary cube such that $x \in \overset{\circ}{Q}$. Then by Proposition 2.15(vi) $Q \subset X$. Let $V \in \mathcal{K}_0(Q)$ be any vertex of Q. Since by Proposition 2.52 Q is edge connected, $Q \subset \mathrm{ecc_X(x)}$ and $Q \subset \mathrm{ecc_X(V)}$. Now if $\mathrm{ecc_X(V) = ecc_X(V_0) = X_0}$, then $x \in X_0$. Otherwise $x \in X_1$. This shows that $X = X_0 \cup X_1$, i.e. X is not connected, a contradiction. □

Corollary 2.57 *If* X *is cubical, then for every* $x \in X$ *its connected component and edge connected component coincide.*

The following lemma will be needed in the proof of the main theorem of this section.

Lemma 2.58 *Assume X is a cubical set and $X_1, X_2, \dots X_n$ are its connected components. If $c_i \in C_k(X_i)$ are k-dimensional chains then*

$$|\sum_{i=1}^{n} c_i| = \bigcup_{i=1}^{n} |c_i|.$$

Proof. The left hand side is contained in the right hand side by Proposition 2.19(iv). To show the opposite inclusion take $x \in \bigcup_{i=1}^{n} |c_i|$. Then for some $i_0 \in \{1, 2, \dots n\}$ there exists a $Q \in \mathcal{K}_k(X_{i_0})$ such that $x \in Q$ and $c_{i_0}(Q) \neq 0$. Since $Q \notin \mathcal{K}_k(X_j)$ for $j \neq i_0$ it must be $c_j(Q) = 0$ for $j \neq i_0$. It follows that

$$\left(\sum_{i=1}^{n} c_i\right)(Q) = c_{i_0}(Q) \neq 0,$$

i.e. $x \in |\sum_{i=1}^{n} c_i|$. $\square$

Finally we are able to prove the main theorem of this section.

Theorem 2.59 *Let X be a cubical set. Then, $H_0(X)$ is a free abelian group. Furthermore, if $\{P_i \mid i = 1, \dots, n\}$ is a collection of vertices in X consisting of one vertex from each connected component of X, then*

$$\{[\widehat{P_i}] \in H_0(X) \mid i = 1, \dots, n\}$$

forms a basis for $H_0(X)$.

Proof. Let $X_i := \mathrm{cc_X(P_i)}$ and let $c \in Z_0(X)$. Since $Z_0(X) = C_0(X)$, there exist integers α_p such that by Proposition 2.53

$$[c] = \sum_{P \in \mathcal{K}_0(X)} \alpha_P [\widehat{P}] = \sum_{i=1}^{n} \sum_{P \in \mathcal{K}_0(X_i)} \alpha_P [\widehat{P}] = \sum_{i=1}^{n} \left(\sum_{P \sim_X P_i} \alpha_P \right) [\widehat{P_i}].$$

This shows that the classes $[\widehat{P_i}]$ generate $H_0(X)$.

It remains to show that the generators are free, i.e.

$$\sum_{i=1}^{n} \alpha_i [\widehat{P_i}] = 0$$

implies that all $\alpha_i = 0$. To do so put $c := \sum_{i=1}^{n} \alpha_i \widehat{P_i}$, $X_i := \mathrm{cc_X(P_i)}$ and select a $b \in C_1(X)$ such that $c = \partial b$. Let $b = \sum_{E \in \mathcal{K}_1(X)} \beta_E \widehat{E}$. Let

$$b_i := \sum_{E \in \mathcal{K}_1(X_i)} \beta_E \widehat{E}.$$

We have

$$\sum_{i=1}^{n} \alpha_i \widehat{P_i} = c = \partial b = \sum_{i=1}^{n} \partial b_i.$$

Therefore

$$0 = \sum_{i=1}^{n} \left(\alpha_i \widehat{P}_i - \partial b_i \right).$$

But

$$|\alpha_i \widehat{P}_i - \partial b_i| \subset X_i,$$

therefore by Lemma 2.58

$$\emptyset = |0| = \bigcup_{i=1}^{n} |\alpha_i \widehat{P}_i - \partial b_i|,$$

which shows that $|\alpha_i \widehat{P}_i - \partial b_i| = \emptyset$, i.e. by Proposition 2.19(i) $\alpha_i \widehat{P}_i = \partial b_i$.

Let $\epsilon : C_0(X) \to \mathbf{Z}$ be the group homomorphism defined by $\epsilon(\widehat{P}) = 1$ for every vertex $P \in X$. Let E be an elementary edge. Then, $\partial \widehat{E} = \widehat{V}_1 - \widehat{V}_0$ where V_0 and V_1 are vertices of E. Observe that

$$\begin{aligned}
\epsilon(\partial \widehat{E}) &= \epsilon(\widehat{V}_1 - \widehat{V}_0) \\
&= \epsilon(\widehat{V}_1) - \epsilon(\widehat{V}_0) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

This implies that $\epsilon(\partial b_i) = 0$ and hence

$$0 = \epsilon(\partial b_i) = \epsilon(\alpha_i \widehat{P}_i) = \alpha_i \epsilon(\widehat{P}_i) = \alpha_i.$$

□

Exercises

2.27 * Show that $\sim_X$ is an equivalence relation.

2.28 Find all minimal edge paths in the unit cube $Q = [0,1]^3$ connecting the vertex $V_0 = (0,0,0)$ to $V_1 = (1,1,1)$. For each edge path present the corresponding chain c such that $\partial c = \widehat{V_1} - \widehat{V_0}$.

2.29 * Prove Proposition 2.52.

2.4 Elementary Collapses

As the reader might have realized by now, even very "simple" cubical sets contain a large number of elementary cubes. We shall now discuss a method that allows us to reduce the number of elementary cubes needed to compute the homology of the set. Recall from Definition 2.7 that elementary cubes are decomposed into faces and proper faces. We now refine these concepts further, but in the context of cubical sets rather than elementary cubes.

Definition 2.60 Let X be a cubical set and let $Q \in \mathcal{K}(X)$. If Q is not a proper face, of some $P \in \mathcal{K}(X)$ then it is a *maximal face* in X. $\mathcal{K}_{\max}(X)$ is the set of maximal faces in X. A face which is a proper face of exactly one elementary cube in X is a *free face in* X.

Example 2.61 Let $X = [0,1] \times [0,1] \times [0,1]$. Then, $\mathcal{K}_0(X) \cup \mathcal{K}_1(X) \cup \mathcal{K}_2(X)$ is the set of proper faces. The set of free faces is given by $\mathcal{K}_2(X)$. For this simple case, $\mathcal{K}_{\max}(X) = \{X\}$.

Given a finite collection $\mathcal{X}$ of elementary cubes in $\mathbf{R}^d$, we use the notation

$$|\mathcal{X}| := \bigcup \mathcal{X}. \tag{2.10}$$

Obviously, $|\mathcal{X}|$ is a cubical set.

Example 2.62 We are referring to the cubical set $X \subset \mathbf{R}^2$ shown in Figure 2.7. The following elementary cubes are free faces

$$\begin{array}{l} [-1] \times [2] \\ [0,1] \times [0], \quad [0,1] \times [1], \quad [0] \times [0,1], \quad [1] \times [0,1] \end{array}$$

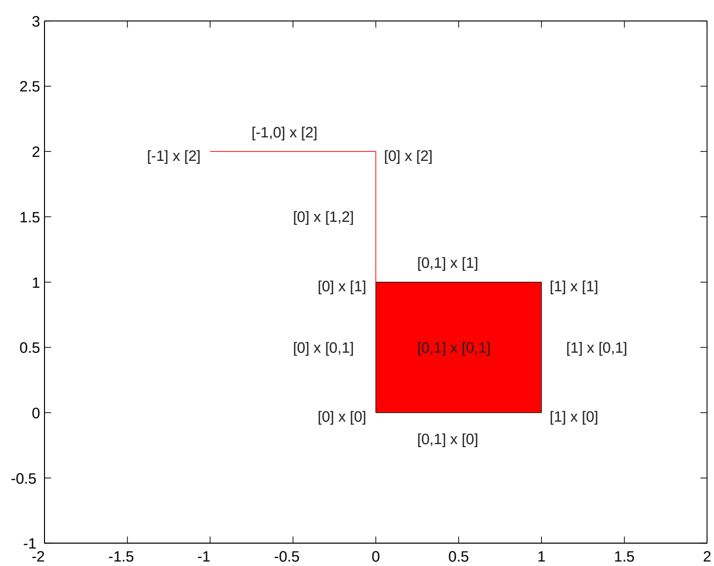


Fig. 2.7. Elementary cubes of $X \subset \mathbf{R}^2$.

Lemma 2.63 *Let X be a cubical set. Let $Q \in \mathcal{K}(X)$ be a free face in X and assume $Q \prec P \in \mathcal{K}(X)$. Then $P \in \mathcal{K}_{\max}(X)$ and* $\dim Q = \dim P - 1$.

Proof. Assume $P \prec R$. Then $Q \prec R$ contradicting the uniqueness of P.

Assume $\dim Q < \dim P - 1$. Then there exists $R \in \mathcal{K}(X)$ different from Q and P such that $Q \prec R \prec P$. □

Definition 2.64 Let Q be a free face in X and let P be the unique cube in $\mathcal{K}(X)$ such that Q is a proper face of P. Let $\mathcal{K}'(X) := \mathcal{K}(X) \setminus \{Q, P\}$. Define

$$X' := \bigcup_{R \in \mathcal{K}'(X)} R.$$

Then X' is a cubical space obtained from X via an *elementary collapse of P by Q*.

Proposition 2.65 *If X' is a cubical space obtained from X via an elementary collapse of P by Q then*

$$\mathcal{K}(X') = \mathcal{K}'(X).$$

Proof. The inclusion $\mathcal{K}'(X) \subset \mathcal{K}(X')$ is obvious. To prove the opposite inclusion assume that there exists an elementary cube $S \in \mathcal{K}(X') \backslash \mathcal{K}'(X)$. It follows that $S \in \{P, Q\}$. Let $x \in \overset{\circ}{S} \subset S \subset X'$. Then $x \in R$ for some $R \in \mathcal{K}'(X)$ and $R \cap \overset{\circ}{S} \supset \{x\} \neq \emptyset$. By Proposition 2.15(vi) $S \subset R$. Since $S \notin \mathcal{K}'(X)$, S is a proper face of $R \in \mathcal{K}'(X) \subset \mathcal{K}(X)$. But neither $S = Q$ nor $S = P$ can be a proper face of such an R, a contradiction. □

Example 2.66 Let $X = [0,1] \times [0,1] \subset \mathbf{R}^2$ (see Figure 2.8). Then

$$\begin{aligned}
\mathcal{K}_2(X) &= \{[0,1] \times [0,1]\} \\
\mathcal{K}_1(X) &= \{[0] \times [0,1], [1] \times [0,1], [0,1] \times [0], [0,1] \times [1]\} \\
\mathcal{K}_0(X) &= \{[0] \times [0], [0] \times [1], [1] \times [0], [1] \times [1]\}
\end{aligned}$$

There are four free faces, the elements of $\mathcal{K}_1(X)$. Let $Q = [0,1] \times [1]$, then $Q \prec P = [0,1] \times [0,1]$. If we let X' be the cubical space obtained from X via the elementary collapse of P by Q, then $X' = [0] \times [0,1] \cup [1] \times [0,1] \cup [0,1] \times [0]$ and

$$\begin{aligned}
\mathcal{K}_1(X') &= \{[0] \times [0,1], [1] \times [0,1], [0,1] \times [0]\} \\
\mathcal{K}_0(X') &= \{[0] \times [0], [0] \times [1], [1] \times [0], [1] \times [1]\}
\end{aligned}$$

Observe that the free faces of X' are different from those of X. In particular, $[0] \times [1]$ and $[1] \times [1]$ are free faces with $[0] \times [1] \prec [0] \times [0,1]$. Let X'' be the space obtained by collapsing $[0] \times [0,1]$ by $[0] \times [1]$. Then,

$$\begin{aligned}
\mathcal{K}_1(X'') &= \{[1] \times [0,1], [0,1] \times [0]\} \\
\mathcal{K}_0(X'') &= \{[0] \times [0], [1] \times [0], [1] \times [1]\}
\end{aligned}$$

On X'' we can now perform an elementary collapse of $[1] \times [0,1]$ by $[1] \times [1]$ to obtain X''' where

$$\mathcal{K}_1(X'') = \{[0,1] \times [0]\}$$
$$\mathcal{K}_0(X'') = \{[0] \times [0], [1] \times [0], \}$$

A final elementary collapse of $[0,1] \times [0]$ by $[1] \times [0]$ results in the single point $X'''' = [0] \times [0]$. Thus, through this procedure we have reduced a 2-cube to a single point.

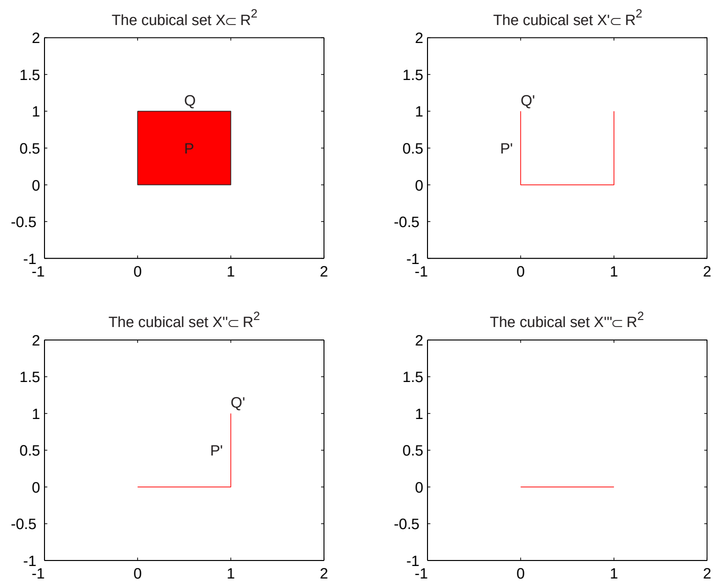
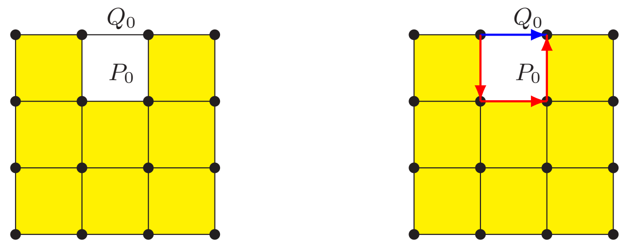


Fig. 2.8. Sequence of Elementary Collapses of $[0,1] \times [0,1] \subset \mathbf{R}^2$.

In the previous example, using elementary collapse, we reduced the cubical complex from that of a unit square containing nine elementary cubes to a cubical complex consisting of a single vertex. We will now prove that two cubical complexes that are related by an elementary collapse have the same homology. Thus, the homology of a square is the same as the homology of a vertex, the latter of course is trivial to compute. We begin with a lemma, which when viewed from a geometrical point of view (see Figure 2.9), is fairly simple.

Lemma 2.67 *Assume X is a cubical set and X' is obtained from X via an elementary collapse of $P_0 \in \mathcal{K}_k(X)$ by $Q_0 \in \mathcal{K}_{k-1}(X)$. Then*

(i)
$$\{c \in C_k(X) \mid \partial c \in C_{k-1}(X')\} \subset C_k(X') \tag{2.11}$$



(i) (ii)

Fig. 2.9. (i) Yellow indicates X' obtained from X by the elementary collapse of $P_0 \in \mathcal{K}_2(X)$ by $Q_0 \in \mathcal{K}_1(X)$. Observe that if $c \in C_2(X)$ and $\partial c \in C_{k-1}(X')$, then the support of c consists entirely of the duals of the yellow squares. (ii) $c \in C_1(X)$ is indicated by the blue arrow. $c' \in C_1(X')$ is indicated by the red arrows. Observe that $c - c' \in B_1(X)$.

(ii) for every $c \in C_{k-1}(X)$ there exists $c' \in C_{k-1}(X')$ such that $c - c' \in B_{k-1}(X)$.

Proof. The proof of (i) begins with some simple observations. First, since P_0 is the unique element of $\mathcal{K}_k(X)$ of which Q_0 is a face, $\langle \partial \widehat{P}, \widehat{Q}_0 \rangle = 0$ if $P \neq P_0$. Similarly, by Exercise 2.19 $\langle \partial \widehat{P_0}, \widehat{Q_0} \rangle = \pm 1$. Now assume that $c \in C_k(X)$ is such that $\partial c \in C_{k-1}(X')$. Then $|\partial c| \subset X'$. In particular $Q_0 \not\subset |\partial c|$ and consequently $\langle \partial c, \widehat{Q}_0 \rangle = 0$. Since $c = \sum_{P \in \mathcal{K}_k(X)} \langle c, \widehat{P} \rangle \widehat{P}$, we have

$$0 = \langle \partial \sum_{P \in \mathcal{K}_k(X)} \langle c, \widehat{P} \rangle \widehat{P}, \widehat{Q}_0 \rangle = \sum_{P \in \mathcal{K}_k(X)} \langle c, \widehat{P} \rangle \langle \partial \widehat{P}, \widehat{Q}_0 \rangle = \pm \langle c, \widehat{P_0} \rangle.$$

Therefore, $c \in C_k(X')$.

To prove (ii) assume that $c \in C_{k-1}(X)$. Let

$$c' := c - \langle c, \widehat{Q}_0 \rangle \langle \partial \widehat{P}_0, \widehat{Q_0} \rangle \partial \widehat{P_0}$$

Then obviously $c - c' = \langle c, \widehat{Q}_0 \rangle \langle \partial \widehat{P}_0, \widehat{Q_0} \rangle \partial \widehat{P_0} \in B_{k-1}(X)$. Since

$$\langle c', \widehat{Q}_0 \rangle = \langle c, \widehat{Q}_0 \rangle - \langle c, \widehat{Q}_0 \rangle \langle \partial \widehat{P}_0, \widehat{Q_0} \rangle^2 = 0,$$

it follows that $c' \in C_{k-1}(X')$. □

Theorem 2.68 *Assume X is a cubical set and X' is obtained from X via an elementary collapse of $P_0 \in \mathcal{K}_k(X)$ by $Q_0 \in \mathcal{K}_{k-1}(X)$. Then*

$$H_*(X') \cong H_*(X).$$

Proof. We begin the proof by making some obvious comparisons between the chains, cycles and boundaries of the two complexes. Since $X' \subset X$, it follows that $C_n(X') \subset C_n(X)$ for all $n \in \mathbf{Z}$. Moreover, by Proposition 2.65 $\mathcal{K}_n(X') = \mathcal{K}_n(X)$ for $n \in \mathbf{Z} \setminus \{k-1, k\}$, therefore

$$C_n(X') = C_n(X) \quad \text{for all } n \in \mathbf{Z} \setminus \{k-1, k\}.$$

Since by (2.8) $Z_n(X) = C_n(X) \cap \ker \partial_n$ for any cubical set X and any $n \in \mathbf{Z}$, we have $Z_n(X') \subset Z_n(X)$ for all $n \in \mathbf{Z}$ and

$$Z_n(X') = Z_n(X) \quad \text{for all } n \in \mathbf{Z} \setminus \{k-1, k\}.$$

Similarly, since by (2.9) $B_n(X) = \partial_{n+1}(C_{n+1}(X))$ for any cubical set X and any $n \in \mathbf{Z}$, we have $B_n(X') \subset B_n(X)$ for all $n \in \mathbf{Z}$ and

$$B_n(X') = B_n(X) \quad \text{for all } n \in \mathbf{Z} \setminus \{k-2, k-1\}.$$

Observe that since the cycles and boundaries are the same,

$$H_k(X') = H_k(X) \quad \text{for all } n \in \mathbf{Z} \setminus \{k-2, k-1, k\}.$$

The only possible difference on the level of the k-chains is in the cycles. We now show that they are the same.

$$\begin{aligned}
Z_k(X) &= C_k(X) \cap \ker \partial_k \\
&= C_k(X) \cap \ker \partial_k \cap \ker \partial_k \\
&\overset{1}{\subset} C_k(X) \cap \{c \in C_k \mid \partial_k c \in C_{k-1}(X')\} \cap \ker \partial_k \\
&\overset{2}{\subset} C_k(X') \cap \ker \partial_k \\
&= Z_k(X').
\end{aligned}$$

Inclusion 1 is due to the fact that $\{0\} \subset C_{k-1}(X')$, while inclusion 2 follows from Lemma 2.67(i). Thus, $H_k(X') = H_k(X)$.

On the level of the $(k-2)$-chains the only possible difference is in the boundaries. We will show they are the same by proving that $B_{k-2}(X) \subset B_{k-2}(X')$. To this end take $b \in B_{k-2}(X)$. Then $b = \partial c$ for some $c \in C_{k-1}(X)$. By Lemma 2.67(ii) there exists $c' \in C_{k-1}(X')$ such that $c - c' \in B_{k-1}(X)$. In particular, for some d,

$$b - \partial c' = \partial(c - c') = \partial^2 d = 0$$

and thus $b = \partial c' \in B_{k-2}(X')$. Therefore we have shown that $Z_n(X) = Z_n(X')$ and $B_n(X) = B_n(X')$ for $n \in \mathbf{Z} \setminus \{k-1\}$. It follows that $H_{k-2}(X') = H_{k-2}(X)$.

It remains to consider the homology groups in dimension $k-1$. In this case the homology groups are not equal, but we will construct an isomorphism. Let $\xi \in H_{k-1}(X')$. Then $\xi = [z]_{X'}$ for some cycle $z \in Z_{k-1}(X')$. We put the subscript X' in $[z]_{X'}$ to emphasize that the homology class is taken in X'. This is important, because $Z_{k-1}(X') \subset Z_{k-1}(X)$, which means that we can also consider the homology class $[z]_X$ of z in $H_{k-1}(X)$. Since $B_{k-1}(X') \subset$

$B_{k-1}(X)$, one easily verifies that if $[z]_{X'} = [z']_{X'}$ for some $z' \in Z_{k-1}(X')$, then $[z]_X = [z']_X$. This means that we have a well defined map $\iota : H_{k-1}(X') \to H_{k-1}(X)$ given by $\iota([z]_{X'}) = [z]_X$. It is straightforward to verify that this map is a homomorphism of groups. To show that it is a monomorphism assume that $[z]_X = 0$ for some $z \in Z_{k-1}(X')$. Then $z = \partial c$ for some $c \in C_k(X)$. It follows from Lemma 2.67(i) that $c \in C_k(X')$, which shows that $[z]_{X'} = 0$. Consequently the map is a monomorphism. To show that it is an epimorphism consider $[z]_X$ for some $z \in Z_{k-1}(X)$. By Lemma 2.67(ii) there exists $c' \in C_{k-1}(X')$ such that $z - c' \in B_{k-1}(X)$. In particular $\partial(z - c') = 0$, i.e. $\partial c' = \partial z = 0$. This shows that $c' \in Z_{k-1}(X')$ and since $\iota([c']_{X'}) = [c']_X = [z]_X$, the map is surjective. □

Corollary 2.69 *Let $Y \subset X$ be cubical sets. Furthermore, assume that Y can be obtained from X via a series of elementary collapses, then*

$$H_*(Y) \cong H_*(X).$$

Elementary collapses were introduced in combinatorial fashion. One began with a cubical complex and removed two specially chosen elementary cubes. In particular, there was no topological motivation provided. This is due to the fact that to provide a complete explanation requires understanding how continuous maps induce linear maps between homology groups; a topic that is covered in Chapter 6. Therefore, with this in mind, we shall limit ourselves to some simple examples that are meant to suggest that elementary collapses are induced by continuous maps called deformation retractions defined as follows.

Definition 2.70 Let $A \subset X$. A *deformation retraction* of X onto A is a continuous map $h : X \times [0,1] \to X$ such that

$$\begin{aligned} h(x,0) &= x \text{ for all } x \in X \\ h(x,1) &\in A \text{ for all } x \in X \\ h(a,1) &= a \text{ for all } a \in A. \end{aligned}$$

If such an h exists, then A is called a *deformation retract* of X. The map h is called a *strong deformation retraction* and the set A a *strong deformation retract* if the third identity is reinforced as follows

$$h(a,s) = a \text{ for all } a \in A \text{ and all } s \in [0,1].$$

Deformation retracts which are not strong deformation retracts only occur in complicated spaces. In our class of cubical sets, all examples of deformation retracts will be strong. The relation of deformation retraction to homology will be discussed in Section 6.5.

Example 2.71 Let $P = [0,1]$ The free faces in P are its vertices $Q_0 = [0]$ and $Q_1 = [1]$. Observe that Q_1 is obtained from P by the elementary collapse of P by Q_0. It is easy to show that the map $h : P \times [0,1] \to P$ defined by

$$h(x, s) := (1 - s)x + s$$

is a strong deformation retraction of P onto Q_1. This construction can be easily carried over to any one dimensional elementary cube and its vertex (this also is a particular case of Exercise 6.17). Thus, in the case of $P \in \mathcal{K}_1$, an elementary collapse corresponds to a deformation retraction.

Example 2.72 Let $P = [0, 1] \times [0, 1]$ and let $Q = [1] \times [0, 1]$. Observe that Q is a free face of P and that $P' = [0] \times [0, 1] \cup [0, 1] \times [0] \cup [0, 1] \times [1]$ is the result of the elementary collapse of P by Q. We shall now show that P' is a strong deformation retract of P.

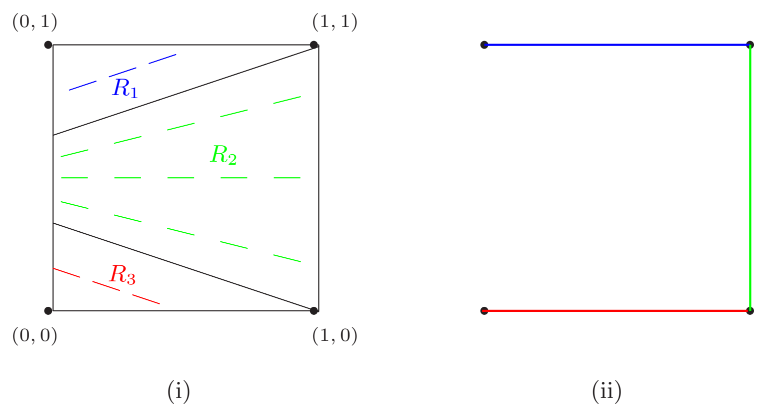


Fig. 2.10. Deformation retraction of a square. (i) The solid lines indicate the regions R_1, R_2, and R_3. The dashed lines indicate the lines along which the deformation occurs. (ii) The image of the collapsed square. The color codes for the images of the three regions.

We begin by decomposing P into three subsets. Let

$$R_1 := \left\{(x, y) \in P \mid y \geq \frac{1}{3}x + \frac{2}{3}\right\}$$

$$R_2 := \left\{(x, y) \in P \mid -\frac{1}{3}x + \frac{1}{3} \leq y \leq \frac{1}{3}x + \frac{2}{3}\right\}$$

$$R_3 := \left\{(x, y) \in P \mid y \leq -\frac{1}{3}x + \frac{1}{3}\right\}.$$

We now provide an alternative description for the elements of R_i.

$$R_1 := \left\{(x,y) \in P \mid y = \frac{1}{3}x + \theta,\ \frac{2}{3} \le \theta \le 1,\ 0 \le x \le 3(1-\theta)\right\}$$

$$R_2 := \left\{(x,y) \in P \mid y = (2\theta - 1)x + \theta,\ \frac{1}{3} \le \theta \le \frac{2}{3},\ 0 \le x \le 1\right\}$$

$$R_3 := \left\{(x,y) \in P \mid y = -\frac{1}{3}x + \theta,\ 0 \le \theta \le \frac{1}{3},\ 0 \le x \le 3\theta\right\}.$$

Consider the map $h : P \times [0,1] \to P$ defined by

$$h((x,y),s) := \begin{cases} (1-s)(x, \frac{1}{3}x + \theta) + s(3(1-\theta), 1) & \text{if } (x,y) \in R_1, \\ (1-s)(x, (2\theta-1)x + \theta) + s(1, 3\theta - 1) & \text{if } (x,y) \in R_2, \\ (1-s)(x, -\frac{1}{3}x + \theta) + s(3\theta, 0) & \text{if } (x,y) \in R_3. \end{cases}$$

It is left as an exercise to show that h is continuous. Observe that $(x,y) \mapsto h((x,y),0)$ is the identity map on P, while $x \mapsto h((x,y),1)$ maps P to P'. Moreover $h((x,y),s) = (x,y)$ for all $(x,y) \in P'$ and all s. Thus h is a strong deformation retraction of P onto P'.

Exercises

2.30 Determine the maximal and free faces of the cubical set presented in the file exC2d.txt in the folder CubTop/Examples by calling the program CubTop with the corresponding functions. Present your output in two bitmap files.

2.31 * Let X be a 2-dimensional cubical subset of $\mathbf{R}^2$. Show that X has at least one free edge.

2.32 Use the elementary collapses to reduce the elementary cube $[0,1]^3$ to its vertex $(0,0,0)$.

2.33 Let X be the solid cubical set discussed in Exercise 2.24. Here is an alternative way of computing the homology of X: Use the elementary collapses of X onto the simple closed curve Γ defined as the union of four line segments $[1,2] \times [1] \times [0]$, $[2] \times [1,2] \times [0]$, $[1,2] \times [2] \times [0]$, $[1] \times [1,2] \times [0]$. Compute the homology of Γ and deduce what is the homology of X.

2.34 Let X be a cubical set in $\mathbf{R}^2$ of dimension 2. In Exercise 2.31 we show that any such set must have a free edge. Prove that X can be reduced to a 1-dimensional cubical set by elementary collapses. Conclude that $H_2(X) = 0$.

2.35 Let X be a cubical set and let $\mathcal{M} \subset \mathcal{K}_{\max}(X)$ be a family of selected maximal faces of X. Put

$$X' := \sum_{R \in \mathcal{K}(X) \setminus \mathcal{M}} R.$$

Show that

$$\mathcal{K}(X') = \mathcal{K}(X) \setminus \mathcal{M}.$$

2.36 Let $P = [0,1] \times [0,1] \times [0,1]$. Then $Q = [0] \times [0,1] \times [0,1]$ is a free face of P. Let P' be obtained via an elementary collapse of P by Q. Construct a deformation retraction of P to P'.

2.5 Acyclic Cubical Spaces

We shall now study a class of important cubical sets; those which have trivial homology, i.e. the homology of a point.

Definition 2.73 A cubical set X is *acyclic* if

$$H_k(X) \approx \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.66 shows that the unitary cube $[0,1]^2$ in $\mathbf{R}^2$ is acyclic and Exercise 2.32 shows that $[0,1]^3$ in $\mathbf{R}^3$ is acyclic. We may conjecture that any elementary cube is acyclic. Although the idea of collapsing cubes onto their faces is very transparent, writing it down precisely requires a lot of work. Therefore we present another proof, based on a special feature of cycles located in elementary cubes.

Let $Q \in \mathcal{K}^d$ be an elementary cube. For some $i \in \{1, 2, \ldots d\}$ let $I_i(Q)$ be nondegenerate, i.e $I_i(Q) = [l, l+1]$, where $l \in \mathbf{Z}$. Fix $k > 0$. The family of k-dimensional faces of Q decomposes into

$$\mathcal{K}_k(Q) = \mathcal{K}([l], k) \cup \mathcal{K}([l, l+1], k) \cup \mathcal{K}([l+1], k),$$

where $\mathcal{K}(\Delta, k) := \{P \in \mathcal{K}_k(Q) \mid \; I_i(P) = \Delta\}$.

Example 2.74 Let $Q = [p, p+1] \times [l, l+1] \times [q]$. Let $i = 2$ and $k = 1$. Then

$$\begin{aligned} \mathcal{K}([l], k) &= \{[p, p+1] \times [l] \times [q]\} \\ \mathcal{K}([l, l+1], k) &= \{[p] \times [l, l+1] \times [q], [p+1] \times [l, l+1] \times [q]\} \\ \mathcal{K}([l+1], k) &= \{[p, p+1] \times [l+1] \times [q]\} \end{aligned}$$

As the following lemma shows, k-cycles in elementary cubes have a nice geometric feature: if $z \in Z_k(Q)$ and $|z|$ contains no elementary cubes projecting onto $[l+1]$ on i-th component, then it contains no elementary cubes projecting onto $[l, l+1]$.

Lemma 2.75 *Assume z is a k-cycle in Q such that $\langle z, \widehat{P} \rangle = 0$ for every $P \in \mathcal{K}([l+1], k)$. Then $\langle z, \widehat{P} \rangle = 0$ for every $P \in \mathcal{K}([l, l+1], k)$.*

Proof. Since z is a chain in Q, we have

$$z = \sum_{P \in \mathcal{K}_k(Q)} \langle z, \widehat{P} \rangle \widehat{P}.$$

Therefore for any $R \in \mathcal{K}_{k-1}(Q)$ we have

$$\langle \partial z, \widehat{R} \rangle = \langle \sum_{P \in \mathcal{K}_k(Q)} \langle z, \widehat{P} \rangle \partial \widehat{P}, \widehat{R} \rangle = \sum_{P \in \mathcal{K}_k(Q)} \langle z, \widehat{P} \rangle \langle \partial \widehat{P}, \widehat{R} \rangle.$$

Since z is a cycle, it follows by our assumption that for any $R \in \mathcal{K}_{k-1}(Q)$

$$0 = \langle \partial z, \widehat{R} \rangle = \sum_{P \in \mathcal{K}([l],k)} \langle z, \widehat{P} \rangle \langle \partial \widehat{P}, \widehat{R} \rangle + \sum_{P \in \mathcal{K}([l,l+1],k)} \langle z, \widehat{P} \rangle \langle \partial \widehat{P}, \widehat{R} \rangle \tag{2.12}$$

Let $P_0 \in \mathcal{K}([l,l+1],k)$ and let R_0 be the elementary cube defined by

$$I_j(R_0) = \begin{cases} [l+1] & \text{if } j = i, \\ I_j(P_0) & \text{otherwise.} \end{cases}$$

Obviously R_0 cannot be a face of P for $P \in \mathcal{K}([l],k)$, hence the first sum in the right hand side of (2.12) disappears for $R = R_0$. Moreover, R_0 is a face of P for $P \in \mathcal{K}([l,l+1],k)$ if and only if $P = P_0$. This means that equation (2.12) reduces for $R = R_0$ to $0 = \langle z, \widehat{P_0} \rangle \langle \partial \widehat{P_0}, \widehat{R_0} \rangle$. Since obviously $\langle \partial \widehat{P_0}, \widehat{R_0} \rangle \neq 0$, we get $\langle z, \widehat{P_0} \rangle = 0$. $\square$

Theorem 2.76 *All elementary cubes are acyclic.*

Proof. Let Q be an elementary cube. Since Q is connected, it follows from Theorem 2.59 that $H_0(Q) = \mathbf{Z}$. Therefore it remains to prove that $H_k(Q) = 0$ for $k > 0$, which is equivalent to showing that every k-cycle in Q is a boundary. We will show this fact by induction in $n := \dim Q$. If $n = 0$ and $k > 0$, then $Z_k(Q) = C_k(Q) = 0 = B_k(Q)$ which implies that $H_k(Q) = 0$.

Therefore, assume that $n > 0$ and $H_k(Q) = 0$ for all elementary cubes of dimension less than n. Since $n > 0$, we can choose some i such that $I_i(Q)$ is nondegenerate. For every $P \in \mathcal{K}([l+1],k)$ let P^* be the elementary cube given by

$$I_j(P^*) := \begin{cases} [l,l+1] & \text{if } j = i, \\ I_j(P_0) & \text{otherwise.} \end{cases}$$

Then obviously P is a face of P^*. Let z be a k-cycle in Q and set

$$c := \sum_{P \in \mathcal{K}([l+1],k)} \langle z, \widehat{P} \rangle \langle \partial \widehat{P^*}, \widehat{P} \rangle \widehat{P^*}$$
$$z' := z - \partial c.$$

For $P \in \mathcal{K}([l+1],k)$ we have

$$\langle \partial c, \widehat{P_0} \rangle = \sum_{P \in \mathcal{K}([l+1],k)} \langle z, \widehat{P} \rangle \langle \partial \widehat{P^*}, \widehat{P} \rangle \langle \partial \widehat{P^*}, \widehat{P_0} \rangle$$

Since $I_i(P^*) = [l,l+1]$ and $I_i(P_0) = [l+1]$, we have $\langle \partial \widehat{P^*}, \widehat{P_0} \rangle \neq 0$ if and only if $P = P_0$, therefore $\langle \partial c, \widehat{P_0} \rangle = \langle z, \widehat{P_0} \rangle$ and $\langle z', \widehat{P_0} \rangle = 0$. It follows from Lemma 2.75 that $|z'| \subset Q^*$, where Q^* is an $(n-1)$-dimensional cube defined by

$$I_j(Q^*) := \begin{cases} [l] & \text{if } j = i, \\ I_j(Q) & \text{otherwise.} \end{cases}$$

By induction assumption $z' = \partial c'$. This shows that $z = \partial(c + c')$, i.e. z is a boundary. $\square$

While Theorem 2.76 shows us that the building blocks of our theory are acylic, it sheds no light on the question how to determine if a given cubical set is acyclic. Theorem 2.78 provides us with such a method. As we shall see in Chapter 6 this is a simple version of a much more general and powerful theorem called the Meyer-Vietoris sequence. Before stating and proving Theorem 2.78 we need the following proposition.

Proposition 2.77 *If $K, L \subset \mathbf{R}^n$ are cubical sets, then*

$$C_k(K \cup L) = C_k(K) + C_k(L).$$

Proof. Since $K \subset K \cup L$, obviously $C_k(K) \subset C_k(K \cup L)$. Similarly $C_k(L) \subset C_k(K \cup L)$. Hence $C_k(K) + C_k(L) \subset C_k(K \cup L)$, because $C_k(K \cup L)$ is a group.

To prove the opposite inclusion let $c \in C_k(K \cup L)$. In terms of the basis elements this can be written as

$$c = \sum_{i=1}^{m} \alpha_i \widehat{Q}_i, \quad \alpha_i \neq 0.$$

Let $A := \{i \mid Q_i \subset K\}$ and $B := \{1, 2, \ldots, m\} \setminus A$. Put $c_1 := \sum_{i \in A} \alpha_i \widehat{Q}_i$, $c_2 := \sum_{i \in B} \alpha_i \widehat{Q}_i$. Obviously $|c_1| \subset K$. Let $i \in B$. Then $Q_i \subset K \cup L$ and $Q_i \not\subset K$. By Proposition 2.15(vi) $\overset{\circ}{Q}_i \cap K = \emptyset$. Therefore $\overset{\circ}{Q}_i \subset L$ and by Proposition 2.15(iv) $Q_i = \operatorname{cl} \overset{\circ}{Q}_i \subset \operatorname{cl} L = L$ Hence $|c_2| \subset L$. It follows that $c = c_1 + c_2 \in C_k(K) + C_k(L)$. $\square$

Theorem 2.78 *Assume $X, Y \subset \mathbf{R}^n$ are cubical sets. If X, Y and $X \cap Y$ are acyclic, then $X \cup Y$ is acyclic.*

Proof. We will first prove that $H_0(X \cup Y) \approx \mathbf{Z}$. By Theorem 2.59 the assumption that X and Y are acyclic implies that X and Y are connected. $X \cap Y$ is acyclic implies that $X \cap Y \neq \emptyset$. Therefore, $X \cup Y$ is connected and hence by Theorem 2.59, $H_0(X \cup Y) \approx \mathbf{Z}$.

Now consider the case of $H_1(X \cup Y)$. Let $z \in Z_1(X \cup Y)$ be a cycle. We need to show that $z \in B_1(X \cup Y)$. By Proposition 2.77, $z = z_X + z_Y$ for some $z_X \in C_1(X)$ and $z_Y \in C_1(Y)$. Since z is a cycle, $\partial z = 0$. Thus,

$$\begin{aligned} 0 &= \partial z \\ &= \partial(z_X + z_Y) \\ &= \partial z_X + \partial z_Y \\ -\partial z_Y &= \partial z_X. \end{aligned}$$

Observe that $-\partial z_Y, \partial z_X \in C_0(X \cap Y) = Z_0(X \cap Y)$. From the assumption of acyclicity, $H_0(X \cap Y)$ is isomorphic to $\mathbf{Z}$.

Let P_0 be a vertex in $X \cap Y$. By Theorem 2.59 $H_0(X \cap Y)$ is generated by $[\widehat{P_0}]_{X\cap Y}$. Let $[\partial z_X]_{X\cap Y}$ denote the homology class of ∂z_X in $H_0(X \cap Y)$. Then, $[\partial z_X]_{X\cap Y} = n[\widehat{P_0}]_{X\cap Y}$ for some $n \in \mathbf{Z}$.

We will now show that $n = 0$. $\partial z_X \in C_0(X\cap Y)$ implies that $\partial z_X = \sum a_i \widehat{P}_i$ where $P_i \in \mathcal{K}_0(X\cap Y)$. By Theorem 2.59, $[\partial z_X]_{X\cap Y} = n[\widehat{P_0}]_{X\cap Y}$ implies that $\sum a_i = n$. Define the group homomorphism $\epsilon : C_0(X \cap Y) \to \mathbf{Z}$ by $\epsilon(\widehat{P}) = 1$ for each $P \in \mathcal{K}_0(X\cap Y)$. Then $\epsilon(\partial\widehat{Q}) = 0$ for any $Q \in \mathcal{K}_1(X\cap Y)$. Therefore, $\epsilon(\partial z_X) = 0$, but

$$\epsilon(\partial z_X) = \sum a_i = n.$$

Hence, $n = 0$.

Since $[\partial z_X]_{X\cap Y} = 0$, there exists $c \in C_1(X \cap Y)$ such that $\partial c = \partial z_X$. Now observe that

$$\partial(-c + z_X) = -\partial c + \partial z_X = 0.$$

Therefore, $-c + z_X \in Z_1(X)$. But $H_1(X) = 0$, which implies that there exists $b_X \in C_2(X)$ such that $\partial b_X = -c + z_X$. The same argument shows that there exits $b_Y \in C_2(X)$ such that $\partial b_Y = c + z_Y$. Finally, observe that $b_X + b_Y \in C_2(X \cup Y)$ and

$$\begin{aligned}
\partial(b_X + b_Y) &= \partial b_X + \partial b_Y \\
&= (-c) + z_X + c + z_Y \\
&= z_X + z_Y \\
&= z.
\end{aligned}$$

Therefore, $z \in B_1(X\cup Y)$ which implies that $[z]_{X\cap Y} = 0$. Thus, $H_1(X\cup Y) = 0$.

We now show that $H_k(X \cup Y) \cong 0$ for all $k > 1$. Let $z \in Z_k(X \cup Y)$ be a cycle. Then by Proposition 2.77, $z = z_X + z_Y$ for some $z_X \in C_k(X)$ and $z_Y \in C_k(Y)$. Since z is a cycle, $\partial z = 0$. Thus,

$$\begin{aligned}
0 &= \partial z \\
&= \partial(z_X + z_Y) \\
&= \partial z_X + \partial z_Y \\
-\partial z_Y &= \partial z_X.
\end{aligned}$$

Of course, this does not imply that $\partial z_X = 0$. However, since $z_Y \in C_k(Y)$ and $z_X \in C_k(X)$ we can conclude that $-\partial z_Y, \partial z_X \in C_{k-1}(X \cap Y)$. Let $b = \partial z_X$. Since

$$\partial b = \partial(\partial z_X) = 0$$

$b \in Z_{k-1}(X \cap Y)$.

Since $X\cap Y$ is acyclic, $H_{k-1}(X\cap Y) = 0$. Therefore, $b \in B_{k-1}(X\cap Y)$. i.e. there exists a $c \in C_k(X\cap Y)$ such that $b = \partial c$. It follows that $z_X - c \in Z_k(X)$

and $z_Y + c \in Z_k(Y)$. By the acyclicity of X and Y there exist $c_X \in C_{k+1}(X)$ and $c_Y \in C_{k+1}(Y)$ such that $z_X - c = \partial c_X$ and $z_Y - c = \partial c_Y$. Therefore

$$z = z_X + z_Y = \partial(c_X + c_Y) \in B_k(X \cup Y).$$

□

Definition 2.79 A *rectangle* is a set of the form $X = [k_1, l_1] \times [k_2, l_2] \times \ldots \times [k_n, l_n] \subset \mathbf{R}^n$, where k_i, l_i are integers and $k_i \leq l_i$.

In particular any rectangle is a cubical set. It is an elementary cube if $k_i = l_i$ or $k_i + 1 = l_i$ for any i.

We leave the proof of the following proposition to the reader.

Proposition 2.80 *A cubical set is convex if and only if it is a rectangle.*

Proof. Exercise 2.37 □

Proposition 2.81 *Any rectangle is acyclic.*

Proof. For $\Delta = [k, l] \subset \mathbf{R}$, an interval with integer endpoints, set

$$\mu(\Delta) := \begin{cases} l - k & \text{if } l > k \\ 1 & \text{otherwise.} \end{cases}$$

Note that Δ is an elementary interval if and only if $\mu(\Delta) = 1$.

Let X be a rectangle, i.e.

$$X = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_d,$$

where $\Delta_i = [k_i, l_i]$ is an interval with integer endpoints. Put

$$\mu(X) := \mu(\Delta_1)\mu(\Delta_2)\ldots\mu(\Delta_d).$$

The proof will proceed by induction on $m := \mu(X)$.

If $m = 1$ then X is an elementary cube, therefore X is acyclic by Theorem 2.76. Thus assume that $m > 1$. Then $\mu(\Delta_{i_0}) = l_{i_0} - k_{i_0} \geq 2$ for some $i_0 \in \{1, 2, \ldots d\}$. Let

$$X_1 := [k_1, l_1] \times \cdots \times [k_{i_0}, k_{i_0} + 1] \times \cdots \times [k_n, l_n]$$

and

$$X_2 := [k_1, l_1] \times \cdots \times [k_{i_0} + 1, l_{i_0}] \times \cdots \times [k_n, l_n].$$

Then, X_1, X_2, and $X_1 \cap X_2$ are rectangles (see Figure 2.11) and

$$\mu(X_1) = \mu(X_1 \cap X_2) = \mu(X)/\mu(\Delta_{i_0}) < m$$
$$\mu(X_2) = \mu(X)\frac{\mu(\Delta_{i_0}) - 1}{\mu(\Delta_{i_0})} < m.$$

Therefore by induction assumption X_1, X_2 and $X_1 \cap X_2$ are acyclic. The result follows now from Theorem 2.78. □

Since rectangles are always the products of intervals they represent a small class of cubical sets. A slightly larger collection of acyclic spaces are the following.

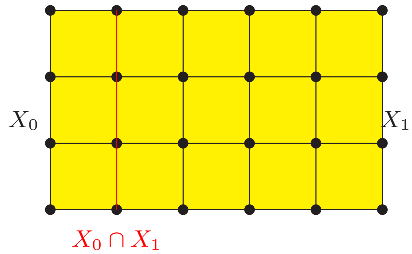


Fig. 2.11. X_0 is the 2-dimensional rectangle on the left. X_1 is the 2-dimensional rectangle on the right. $X_0 \cap X_1$ is the 1-dimensional rectangle.

Definition 2.82 A cubical set $X \subset \mathbf{R}^d$ is *star shaped* with respect to a point $x \in \mathbf{Z}^d$ if X is the union of a finite number of rectangles each of which contains the point x.

Proposition 2.83 *Let X_i, $i = 1, \ldots, n$ be a collection of star shaped sets with respect to the same point x. Then,*

$$\bigcup_{i=1}^{n} X_i \quad \text{and} \quad \bigcap_{i=1}^{n} X_i$$

are star shaped.

Proof. Since X_i is star shaped it can be written as $X_i = \cup R_{i,j}$ where $R_{i,j}$ is a rectangle and $x \in R_{i,j}$. Thus, if $X = \cup_i X_i$, then $X = \cup_{i,j} R_{i,j}$ and hence it is star shaped.

So assume that $X = \cap_i X_i$. Then

$$\begin{aligned} X &= \bigcap_i X_i \\ &= \bigcap_i \left(\bigcup_j R_{i,j} \right) \\ &= \bigcup_j \left(\bigcap_i R_{i,j} \right). \end{aligned}$$

But, since $x \in R_{i,j}$ for every i, j, for each j the set $\bigcap_i R_{i,j}$ is a rectangle and contains x. Again, this means that X is star shaped. □

Proposition 2.84 *Every star shaped set is acyclic.*

Proof. Let X be a star shaped cubical set. Then, $X = \bigcup_{i=1}^{n} R_i$ where each R_i is a rectangle and there exists $x \in X$ such that $x \in R_i$ for all $i = 1, \ldots n$. The proof is by induction on n.

If $n = 1$, then X is a rectangle and hence by Proposition 2.81 is acyclic.

So assume that every star shaped cubical set which can be written as the union of $n-1$ rectangles containing the same point is acyclic. Let $Y = \bigcup_{i=1}^{n-1} R_i$. Then by the induction hypothesis, Y is acyclic. R_n is a rectangle and hence by Proposition 2.81 it is acyclic. Furthermore, $R_i \cap R_n$ is a rectangle for each $i = 1, \ldots, n-1$ and

$$Y \cap R_n = \bigcap_{i=1}^{n-1} (R_i \cap R_n).$$

Therefore, $Y \cap R_n$ is a star shaped set which can be written in terms of $n-1$ rectangles. By the induction hypothesis it is also acyclic. Therefore, by Theorem 2.78, X is acyclic. □

The following simple result will be used later.

Proposition 2.85 *Assume that $\mathcal{Q}$ is a family of rectangles in $\mathbf{R}^d$ such that the intersection of any two of them is non-empty. Then $\bigcap \mathcal{Q}$ is non-empty.*

Proof. First consider the case when $d = 1$. Then rectangles become intervals. Let a denote the supremum of the set of left endpoints of the intervals and let b denote the infimum of the set of right endpoints. We cannot have $b < a$, because then one can find two disjoint intervals in the family. Therefore $\emptyset \neq [a, b] \subset \bigcap \mathcal{Q}$.

If $d > 1$ then each rectangle is a Cartesian product of intervals, the intersection of all rectangles is the Cartesian product of the intersections of the corresponding intervals, and the conclusion follows from the previous case. □

Exercises

2.37 Prove Proposition 2.80.

2.38 Give an example where X and Y are acyclic cubical sets, but $X \cup Y$ is not acyclic.

2.39 Consider the capital letter **H** as a 3-dimensional cubical complex. Compute its homology.

2.6 Homology of Abstract Chain Complexes

We now turn to a purely algebraic description of homology groups. The cubical chain complex is a particular case of what we present below.

Definition 2.86 A *chain complex* $\mathcal{C} = \{C_k, \partial_k\}_{k \in \mathbf{Z}}$ consists of abelian groups C_k, called *chains*, and homomorphisms $\partial_k : C_k \to C_{k-1}$, called *boundary operators*, such that

$$\partial_k \circ \partial_{k+1} = 0. \tag{2.13}$$

$\mathcal{C}$ is a *free chain complex* if C_k is free for all $k \in \mathbf{Z}$. The *cycles* of $\mathcal{C}$ are the subgroups

$$Z_k := \ker \partial_k$$

while the *boundaries* are the subgroups

$$B_k := \operatorname{im} \partial_{k+1}.$$

Observe that (2.13) implies that

$$\operatorname{im} \partial_{k+1} \subset \ker \partial_k \tag{2.14}$$

and hence the following definition makes sense.

Definition 2.87 The k-th *homology group* of the chain complex $\mathcal{C}$ is

$$H_k(\mathcal{C}) := Z_k / B_k.$$

The *homology* of $\mathcal{C}$ is the sequence

$$H_*(\mathcal{C}) := \{H_k(\mathcal{C})\}_{k \in \mathbf{Z}} .$$

Observe that this is a purely algebraic definition. The inclusion 2.14 implies that for each $z \in Z_k$,

$$\partial_k(z + B_k) = \partial_k z + \partial_k(B_k) = 0 + \partial_k(\partial_{k+1}(C_{k+1})) = 0$$

so ∂_k induces the trivial homomorphism

$$\bar{\partial}_k : H_k(\mathcal{C}) \to H_{k-1}(\mathcal{C}) \ , \ \bar{\partial}_k = 0$$

on quotient groups. Thus the homology sequence $H_*(\mathcal{C})$ may also be viewed as a chain complex called *homology complex* of $\mathcal{C}$ with the trivial boundary operators. Conversely, if in some chain complex $\mathcal{C}$ we have $\partial_k = 0$ for all n, then $H_*(\mathcal{C}) = \mathcal{C}$.

Definition 2.88 $\mathcal{C}$ is a *finitely generated free chain complex* if:

1. each C_k is a finitely generated free abelian group,
2. $C_k = 0$ for all but finitely many k.

In this book we are only concerned with finitely generated free chain complexes, however it should be mentioned that we have already seen an example of an infinitely generated free chain complex $\mathcal{C}^d$ with groups C_k^d of chains of $\mathbf{R}^d$ and boundary operators $\partial_k : C_k^d \to C_{k-1}^d$.

Example 2.89 Consider the cubical set consisting of two vertices P and Q. The cubical chain complex $\mathcal{C}(X)$ has only one nontrivial group

$$C_0(X) = \mathbf{Z}\widehat{P} \oplus \mathbf{Z}\widehat{Q} \cong \mathbf{Z}^2.$$

All boundary maps are necessarily 0. Consider now an abstract chain complex $\mathcal{C}$ given by

$$\mathcal{C}_k := \begin{cases} C_0(X) & \text{if } k = 0 \\ \mathbf{Z} & \text{if } k = -1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\partial_k := 0$ if $k \neq 0$. The only nontrivial boundary map $\partial_0 : C_0 \to C_{-1}$ is defined on the generators as follows

$$\partial_0 \widehat{P} = \partial_0 \widehat{P} := 1$$

It is clear that $\operatorname{im} \partial_0 = \mathbf{Z} = C_{-1} = \ker \partial_{-1}$, therefore

$$H_{-1}(\mathcal{C}) = 0$$

We leave as exercise the verification that

$$\ker \partial_0 = \mathbf{Z}(\widehat{P} - \widehat{Q}),$$

and, because $\partial_1 = 0$, we have

$$H_0(\mathcal{C}) = \ker \partial_0 = \mathbf{Z}(\widehat{P} - \widehat{Q}) \cong \mathbf{Z}.$$

The remaining homology groups are trivial. This is a special case of reduced homology, which will be introduced in the next section.

Example 2.90 Define an abstract chain complex $\mathcal{C}$ as follows. The only nontrivial groups C_k of $\mathcal{C}$ are in dimensions $k = 0, 1, 2$. Let

$$C_0 := \mathbf{Z}v \cong \mathbf{Z},$$

$$C_1 := \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \cong \mathbf{Z}^2,$$

$$C_2 := \mathbf{Z}g \cong \mathbf{Z},$$

where v, e_1, e_2, and g are some fixed generators. Define $\partial_1 : C_1 \to C_0$ to be zero. Define $\partial_2 : C_2 \to C_1$ on the generator g by

$$\partial_2 g := 2e_1.$$

The remaining boundary maps must be zero because $C_k = 0$ for all $k \notin \{0, 1, 2\}$. It is easily seen that

$$H_0(\mathcal{C}) = C_0 = \mathbf{Z}v \cong \mathbf{Z}.$$

Computing the group H_1 some acquaintance with abelian groups (see Chapter 13 and Section 3.4).

$$H_1(\mathcal{C}) = \frac{\mathbf{Z}e_1 \oplus \mathbf{Z}e_2}{\mathbf{Z}2e_1} = \frac{\mathbf{Z}e_1}{2\mathbf{Z}e_1} \oplus \mathbf{Z}e_2 \cong Z_2 \oplus Z.$$

Finally, since $\partial_3 = 0$ and $\ker \partial_2 = 0$, we have

$$H_2(\mathcal{C}) = 0.$$

Definition 2.91 Let $\mathcal{C} = \{C_n, \partial_n\}$ be a chain complex. A chain complex $\mathcal{D} = \{D_n, \partial'_n\}$ is a *chain subcomplex* of $\mathcal{C}$ if:

1. D_n is a subgroup of C_n for all $n \in \mathbf{Z}$.
2. $\partial'_n = \partial_n \mid_{D_n}$.

The condition that $\partial'_n = \partial_n \mid_{D_n}$ indicates the boundary operator of a chain subcomplex is just the boundary operator of the larger complex restricted in its domain. For this reason and to simplify the notation we shall let $\partial' = \partial$.

Example 2.92 The following statements are immediate consequences of the definition of cubical boundary operator.

(a) Let $X \subset Y$ be cubical sets. Then $\mathcal{C}(X)$ is a chain subcomplex of $\mathcal{C}(Y)$.
(b) Let $X \subset \mathbf{R}^d$ be a cubical set. Then $\mathcal{C}(X)$ is a chain subcomplex of $\mathcal{C}(\mathbf{R}^d) := \{C^d_k\}$.

Exercises

2.40 Let $\mathcal{C}$ be the abstract chain complex discussed in Example 2.90.

(a) Compute its homology modulo 2.
(b) Compute its homology modulo 3.
(c) Compute its homology modulo p where p is a prime number.

2.41 Let $\mathcal{C} = \{C_k, \partial_k\}$ be a chain complex and let $\mathcal{D} = \{D_k, \partial'_k\}$ be a chain subcomplex. Consider a new complex whose chain groups are quotient groups C_k/D_k and whose boundary maps

$$\bar{\partial}_k : C_k/D_k \to C_{k+1}/D_{k+1}$$

are given by

$$[c + D_k] \mapsto [\partial_k c + D_{k-1}].$$

Prove that $\bar{\partial}_k$ is well defined and that it is a boundary map.
This new complex is called the *relative chain complex*. The *relative n-cycles* are $Z_k(\mathcal{C}, \mathcal{D}) := \ker \bar{\partial}_k$. The relative *$n$-boundaries* are $B_k(\mathcal{C}, \mathcal{D}) := \ker \bar{\partial}_{k+1}$. The *relative homology groups* are

$$H_k(\mathcal{C}, \mathcal{D}) := Z_k(\mathcal{C}, \mathcal{D})/B_k(\mathcal{C}, \mathcal{D}).$$

(This topic will be discussed in greater detail in Chapter 9).

2.7 Reduced Homology

In the proofs of Theorem 2.59 and Theorem 2.78 we used a specific group homomorphism to deal with the fact that the 0-th homology group was isomorphic to $\mathbf{Z}$. In mathematics seeing a particular trick being employed to overcome a technicality in different contexts suggests the possibility of a general procedure to take care of the problem. In Theorem 2.78 we had to consider three cases. Essentially this was caused by the fact that acyclicity means the homology is $\mathbf{Z}$ in the dimension zero and zero in the higher dimensions. We would need only one case if acyclicity meant the homology being zero in every dimension. However, we know that the homology in dimension zero counts the number of connected components, i.e. it must be non-zero. We can therefore, ask the following question: Is there a different homology theory such that in the previous two examples we would have trivial 0-th level homology?

Hopefully, this question does not seem too strange. We spent most of Chapter 1 motivating the homology theory that we are using and as we did so we had to make choices of how to define our algebraic structures. From a purely algebraic point of view, given $\mathcal{K}(X)$ all we need in order to define homology groups is a chain complex $\{C_k(X), \partial_k\}_{k\in\mathbf{Z}}$. This means that if we change our chain complex, then we will have a new homology theory. The trick we employed involved the group homomorphism $\epsilon : C_0(X) \to \mathbf{Z}$ defined by sending each elementary cubical chain to 1. Furthermore, we showed in each case that $\epsilon \circ \partial_1 = 0$, which means that

$$\operatorname{im} \partial_1 \subset \ker \epsilon.$$

It is with this in mind that we introduce the following definition.

Definition 2.93 Let X be a cubical set. The *augmented cubical chain complex* of X is given by $\{\tilde{C}_k(X), \tilde{\partial}_k\}_{k\in\mathbf{Z}}$ where

$$\tilde{C}_k(X) = \begin{cases} \mathbf{Z} & \text{if } k = -1, \\ C_k(X) & \text{otherwise,} \end{cases}$$

and

$$\tilde{\partial}_k := \begin{cases} \epsilon & \text{if } k = 0, \\ \partial_k & \text{otherwise.} \end{cases}$$

It is left as an exercise to show that the augmented cubical chain complex is an abstract chain complex in the sense of Definition 2.86. The added chain group $C_{-1} = \mathbf{Z}$ in the dimension -1 seems not to carry any geometric information. Nevertheless, it may be interpreted as follows. Since vertices have no faces, we have defined $C_{-1}(X)$ to be 0. But we may also adopt a convention that the empty set $\emptyset$ is the face of any vertex. Hence we define $\tilde{C}_{-1}$ to be the free abelian group generated by the singleton $\widehat{\{\emptyset\}}$ which is isomorphic to $\mathbf{Z}$ with $\widehat{\{\emptyset\}}$ corresponding to 1. The boundary of any dual vertex $\widehat{P}$ is $\tilde{\partial}_0(\widehat{P}) = \widehat{\{\emptyset\}}$ which precisely matches with the definition of ϵ.

Augmenting the chain complex $\mathcal{C}(X)$ by the group C_{-1} actually leads to a reduction of the 0-dimensional homology group. Indeed, the 0-cycles of $C_0(X)$ are all 0-chains while the 0-cycles of $\tilde{C}_0(X)$ are only those which are in $\ker \epsilon$. This observation motivates the following terminology.

Definition 2.94 The homology groups $H_k(\tilde{\mathcal{C}}(X))$ are the *reduced homology groups* of X and are denoted by

$$\tilde{H}_k(X).$$

A chain $z \in C_k(X)$ which is a cycle in $\tilde{Z}_k$ is a *reduced cycle* in $\mathcal{C}(X)$. The homology class of a reduced cycle z with respect to the reduced homology is denoted by $[z]_\sim$.

The following theorem indicates the relationship between the two homology groups we now have at our disposal.

Theorem 2.95 *Let X be a cubical set. $\tilde{H}_0(X)$ is a free abelian group and*

$$H_k(X) \approx \begin{cases} \tilde{H}_0(X) \oplus \mathbf{Z} & \text{for } k = 0 \\ \tilde{H}_k(X) & \text{otherwise.} \end{cases}$$

Furthermore, if $\{P_i \mid i = 0, \ldots, n\}$ is a collection of vertices in X consisting of one vertex from each connected component of X, then

$$\{[P_i - P_0]_\sim \in \tilde{H}_0(X) \mid i = 1, \ldots, n\} \tag{2.15}$$

forms a basis for $\tilde{H}_0(X)$.

Proof. Notice that since we change the boundary operator only on the zero level, obviously $\tilde{H}_k(X) = H_k(X)$ for $k \geq 1$. Therefore it is enough to prove (2.15). We begin with showing that $[P_i - P_0]_\sim$ for $i = 1, 2, \ldots n$ are linearly independent. Assume

$$\sum_{i=1}^{n} \alpha_i [\widehat{P_i} - \widehat{P_0}]_\sim = 0.$$

Then there exists a chain $c \in \tilde{C}_k(X) = C_k(X)$ such that

$$\sum_{i=1}^{n} \alpha_i (\widehat{P_i} - \widehat{P_0}) = \partial c,$$

which can be rewritten as

$$\sum_{i=1}^{n} \alpha_i (\widehat{P_i}) - \left(\sum_{i=1}^{n} \alpha_i \right) \widehat{P_0} = \partial c.$$

Taking regular homology classes we obtain

$$\sum_{i=1}^{n} \alpha_i [\widehat{P_i}] - \left(\sum_{i=1}^{n} \alpha_i \right) [\widehat{P_0}] = 0.$$

But by Theorem 2.59 the homology classes $[\widehat{P_i}]$ for $i = 0, 1, \ldots n$ constitute the basis of $H_0(X)$. Therefore $\alpha_i = 0$ for $i = 1, 2, \ldots n$.

It remains to be shown that $[P_i - P_0]_\sim$ for $i = 1, 2, \ldots n$ generate $\tilde{H}_0(X)$. Let c be a reduced cycle in dimension 0. Then, in particular $c \in C_0(X)$ and by Theorem 2.59 there exists

$$c' = \sum_{i=0}^{n} \alpha_i \widehat{P_i}$$

such that $[c] = [c'] \in H_0(X)$. Therefore, there exists $b \in C_1(X)$ such that $c = c' + \partial_1 b$.

Since z is a reduced cycle, $\epsilon(c) = 0$. On the other hand

$$\begin{aligned}
\epsilon(c) &= \epsilon(c' + \partial_1 b) \\
&= \epsilon(c') + \epsilon(\partial_1 b) \\
&= \epsilon(\sum_{i=0}^{n} \alpha_i \widehat{P_i}) \\
&= \sum_{i=0}^{n} \alpha_i.
\end{aligned}$$

Therefore $\sum_{i=0}^{n} \alpha_i = 0$, which shows that c' is a reduced cycle too. Then, $0 = -\sum_{i=0}^{n} \alpha_i \widehat{P_0}$ and we can write

$$\begin{aligned}
c' &= \sum_{i=0}^{n} \alpha_i \widehat{P_i} - \sum_{i=0}^{n} \alpha_i \widehat{P_0} \\
&= \sum_{i=0}^{n} \alpha_i (\widehat{P_i} - \widehat{P_0}).
\end{aligned}$$

So, using the fact that $c - c'$ is a boundary, on the level of reduced homology we obtain

$$[c]_\sim = [c']_\sim = \sum_{i=0}^{n} \alpha_i [\widehat{P_i} - \widehat{P_0}]_\sim.$$

□

This theorem allows us to give an alternative characterization of acyclic spaces.

Corollary 2.96 *Let X be a nonempty cubical set. Then X is acyclic if and only if*

$$\tilde{H}_*(X) = 0.$$

Exercises

2.42 Verify that the augmented cubical chain complex $\tilde{\mathcal{C}}(X)$ is indeed an abstract chain complex in the sense of Definition 2.86.

3

Computing Homology Groups

In light of the discussion of the previous chapter, given a cubical set X we know that its homology groups $H_*(X)$ are well defined. We also computed $H_*(X)$ for some simple examples and discussed the method of elementary collapses, which can be used in special cases to compute these groups. In this chapter we want to go further and argue that the homology groups of any cubical set are computable; that is, there is an algorithm that produces $H_*(X)$ for any given cubical set X. To be properly defined an algorithm should have a well defined set of inputs and a well defined set of outputs. In the case of homology groups of a cubical set X the input is the list of all elements of $\mathcal{K}_{\max}(X)$. This is a finite set and therefore, in principle, it can be entered into a computer. What the output should be may not be so clear at this moment. Obviously, we want to obtain a set of abelian groups. However, for this to be of use we need to know that we can present these groups in a finite and recognizable form. In particular, if X and Y are two cubical sets, it is desirable that our algorithm outputs $H_*(X)$ and $H_*(Y)$ in such a way that it is evident whether or not they are isomorphic. With this in mind, by the end of this chapter we will prove the following result.

Corollary 3.1 *Any finitely generated abelian group G is isomorphic to a group of the form:*

$$\mathbf{Z}^r \oplus \mathbf{Z}_{b_1} \oplus \mathbf{Z}_{b_2} \oplus \cdots \mathbf{Z}_{b_k}, \tag{3.1}$$

where r is a nonnegative integer, Z_b denotes the group of integers modulo b, $b_i > 1$ provided $k > 0$, and b_i divides b_{i+1} for $i \in \{1, 2, \ldots k-1\}$ provided $k > 1$. The numbers r and $b_1, b_2, \ldots, b_k$ are uniquely determined by G.

Clearly, $\mathbf{Z}^r$ is a free abelian group while $\mathbf{Z}_{b_i}$ are finite cyclic groups. However, what is important from the point of view of classification is that up to isomorphism every finitely generated abelian group is exactly determined by a unique finite set of numbers $\{r, b_1, b_2, \ldots, b_k\}$. The number r is the rank of F and is called the *Betti number* of G and the numbers $b_1, b_2, \ldots, b_k$ are called the *torsion coefficients* of G.

This result is a corollary of the *Decomposition Theorem of Abelian Groups* (Theorem 3.58). We shall prove this theorem by constructing Algorithm 3.61 which, in fact, produces generators of the subgroups $\mathbf{Z}^r$ and $\mathbf{Z}_{b_i}$. While this is obviously a stronger result than Corollary 3.1, for the purpose of classifying homology groups this additional information is not needed. We will, however, make use of this data in Chapter 7 when we discuss maps between homology groups.

As was just mentioned, in this chapter we will develop an algorithm that takes $\mathcal{K}_{\max}(X)$ of a cubical set X as input and outputs generators of homology groups and the integers $\{r, b_1, b_2, \ldots, b_k\}$. This might surprise the reader who has carefully studied the table of contents, since much of the next chapter is also devoted to developing algorithms for computing homology. The justification of this is simple. There are two sides to computability - finding an algorithm and finding an efficient algorithm. In this chapter we produce an algorithm that computes homology groups. However from the point of view of computations it is not practical because it is not efficient. The algorithms in Chapter 4 are used to reduce the size of the chain complex so that the results presented here can be effectively applied.

3.1 Matrix Algebra over Z

Before we begin to develop the promised algorithms we need a careful discussion of matrix algebra over the integers. As we already mentioned, in the case of arbitrary finitely generated free abelian groups there is no natural choice of a basis. To represent a homomorphism as a matrix we must always explicitly state which bases are chosen, and, in fact, we will soon see the benefits of changing the bases. In particular, the canonical basis of the group of k-chains of a cubical set X consists of all elementary k-chains. However, in order to find a basis for the group of k-cycles in X we have to choose another basis in $C_k(X)$. Therefore, if for no other reason than to keep track of the boundary operator it is important to be able to tell how the matrix of a homomorphism changes when we change bases. To do so, assume G is a free, finitely generated, abelian group and $V = \{v_1, v_2, \ldots, v_n\}$ is a basis in G. Consider the homomorphism $\xi_V : G \to \mathbf{Z}^n$, defined on the basis V by

$$\xi_V(v_j) = \mathbf{e}_j \ . \tag{3.2}$$

One can easily check that ξ_V is the inverse of the isomorphism considered in Example 13.66. It is called the *coordinate isomorphism associated with the basis* V. The name is justified by the fact, that if $g = \sum_{i=1}^n x_i v_i$ is an element of G expressed as the linear combination of the elements of the basis, then

$$\xi_V(g) = \mathbf{x} := (x_1, x_2, \ldots x_n) \in \mathbf{Z}^n \ ,$$

i.e. the isomorphism ξ_V provides the coordinates of g in the basis V.

Example 3.2 Let $G := C_0(\{0,1\})$ be the group of 0-chains of the cubical two-point set $X := \{0,1\}$. The canonical basis of G consists of the elementary dual vertices

$$v_1 := \widehat{[0]},\ v_2 := \widehat{[1]}.$$

Any $g \in G$ can be expressed as $g = x_1v_1 + x_2v_2$ with unique $x_1, x_2 \in \mathbf{Z}$. Thus the coordinate isomorphism associated with this basis is given by

$$\xi_V(g) := (x_1, x_2) \in \mathbf{Z}^2.$$

Let $V' := \{g'_1, g'_2, \ldots g'_n\}$ be another basis in G and let the coordinates of g in this basis be given by $\mathbf{x}' := \xi_{V'}(g)$. Then

$$\mathbf{x}' = \xi_{V'}(g) = \xi_{V'}(\xi_V^{-1}(\mathbf{x}))\ ,$$

i.e. the coordinates in the new basis may be computed from the coordinates in the old basis by multiplying the coordinates $\mathbf{x}$ of g in the old basis by the matrix $A_{\xi_{V'}\xi_V^{-1}}$ of the isomorphism $\xi_{V'}\xi_V^{-1} : \mathbf{Z}^n \to \mathbf{Z}^n$ (compare Section 13.3.3). This matrix is called the *change of coordinates matrix from V to V'*. Obviously the matrix $A_{\xi_V\xi_{V'}^{-1}}$, the change of coordinates matrix from V' to V is the inverse of the change of coordinates matrix $A_{\xi_{V'}\xi_V^{-1}}$ from V to V'. When it is clear from the context what are the old basis and the new basis, we will just speak about the change of coordinates matrix and the inverse change of coordinates matrix.

The next example explains how to obtain these matrices.

Example 3.3 Let G and $g \in G$ be as in the previous example and consider the new basis $V' = \{v'_1, v'_2\}$ in G given by

$$v'_1 := v_1,\ v'_2 = v_2 - v_1.$$

The inverse formulas are

$$v_1 = v'_1,\ v_2 = v'_1 + v'_2$$

so $g = x_1v_1 + x_2v_2$ can be written in terms of the new basis as

$$g = x_1v'_1 + x_2(v'_1 + v'_2) = (x_1 + x_2)v'_1 + x_2v'_2.$$

Thus $\mathbf{x}' = (x'_1, x'_2) = \xi_{V'}(g) = (x_1 + x_2, x_2)$ so the change of coordinates is given by

$$A_{\xi_{V'}\xi_V^{-1}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that the columns of this matrix represent coordinates of the old basis elements expressed in terms of the new basis. If we investigate its inverse change of coordinates matrix

$$A_{\xi_V \xi_{V'}^{-1}} = A^{-1}_{\xi_{V'} \xi_V^{-1}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

we immediately notice that the columns are the coordinates of the new basis elements expressed in terms of the original basis. This observation will soon be generalized to any change of basis in $\mathbf{Z}^n$.

Assume now that H is another free, finitely generated, abelian group and $W = \{w_1, w_2, \ldots w_m\}$ is a basis of H. Let $f : G \to H$ be a homomorphism of groups and let $h = f(g)$ for some fixed $g \in G$ and $h \in H$. If $\mathbf{x} := \xi_V(g)$ and $\mathbf{y} := \xi_W(h)$ are the coordinates of g and h respectively in bases V and W then

$$\mathbf{y} = \xi_W(f(g)) = \xi_W f \xi_V^{-1}(\mathbf{x}).$$

Since $\xi_W f \xi_V^{-1} : \mathbf{Z}^n \to \mathbf{Z}^m$, the latter equation may be written in terms of the canonical matrix of $\xi_W f \xi_V^{-1}$ as

$$\mathbf{y} = A_{\xi_W f \xi_V^{-1}} \mathbf{x}$$

Therefore the matrix B of f in the bases V and W equals the canonical matrix of $\xi_W f \xi_V^{-1}$, i.e.

$$B = A_{\xi_W f \xi_V^{-1}} .$$

Assume that $V' := \{v'_1, v'_2, \ldots v'_n\}$ is another basis in G and $W' := \{w'_1, w'_2, \ldots w'_n\}$ is another basis in H.

Proposition 3.4 *Let $f : G \to H$ be a homomorphism and let B and B' denote the matrix of f respectively in the bases V, W and V', W'. Then*

$$B' = A_{\xi_{W'} \xi_W^{-1}} B A_{\xi_V \xi_{V'}^{-1}}$$

Proof. We have (compare (13.30))

$$B' = A_{\xi_{W'} f \xi_{V'}^{-1}} = A_{\xi_{W'} \xi_W^{-1} \xi_W f \xi_V^{-1} \xi_V \xi_{V'}^{-1}} =$$
$$A_{\xi_{W'} \xi_W^{-1}} A_{\xi_W f \xi_V^{-1}} A_{\xi_V \xi_{V'}^{-1}} = A_{\xi_{W'} \xi_W^{-1}} B A_{\xi_V \xi_{V'}^{-1}}$$

□

The above proposition shows that in order to obtain the matrix of a homomorphism in a new basis we multiply the matrix of this homomorphism in the old basis on left by the change of coordinates matrix from V' to V and on right by the change of coordinates matrix from W to W'.

An important special case is the case of subgroups of $\mathbf{Z}^n$. First observe that if $V := \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \subset \mathbf{Z}^m$ is a sequence of elements of $\mathbf{Z}^m$ and we treat the elements of V as column vectors, then we may identify V with the matrix

$$[\mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \; \mathbf{v}_n] \in \mathrm{M}_{m,n}(\mathbf{Z}).$$

It is straightforward to verify that in this case

$$V\mathbf{e}_i = \mathbf{v}_i. \tag{3.3}$$

Proposition 3.5 *The columns* $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ *of an* $n \times n$ *integer matrix* V *constitute a basis of* $\mathbf{Z}^n$ *if and only if* V *is* **Z***-invertible. In this case*

$$A_{\xi_V} = V^{-1}. \tag{3.4}$$

Proof. Assume that the columns of V constitute a basis in $\mathbf{Z}^n$. Since the coordinate isomorphism $\xi_V : \mathbf{Z}^n \to \mathbf{Z}^n$ satisfies $\xi_V(\mathbf{v}_i) = \mathbf{e}_i$, we see from (3.3) that A_{ξ_V} is the inverse of V. Therefore V is **Z**-invertible and property (3.4) is satisfied.

It remains to prove that if V is **Z**-invertible then the columns of V constitute a basis of $\mathbf{Z}^n$. Let W denote the inverse of V. To show that the columns are linearly independent assume that

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = 0.$$

Then

$$0 = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \sum_{i=1}^{n} \alpha_i V \mathbf{e}_i = V \left(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i \right)$$

and

$$0 = W0 = WV \left(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i \right) = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i.$$

Therefore all $\alpha_i = 0$, because the canonical vectors $\mathbf{e}_i$ are linearly independent.

To show that the columns generate $\mathbf{Z}^n$ take $\mathbf{y} \in \mathbf{Z}^n$. Since W is an integer matrix, $\mathbf{x} := (x_1, x_2, \dots x_n) := W\mathbf{y} \in \mathbf{Z}^n$. Then

$$\mathbf{y} = V\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i.$$

□

The relationship between a change of coordinates and a change of basis in $\mathbf{Z}^n$ can also be explained by turning the reasoning around: Given a change of coordinates in $\mathbf{Z}^n$, what is the change of basis associated to it? It is very often more convenient to express old coordinates as a function of the new ones rather than the new as a function of the old. So suppose we have a change of coordinates formula

$$\mathbf{x} = Q\mathbf{u} \tag{3.5}$$

where $\mathbf{x}$ is the original coordinate vector with respect to the canonical basis in $\mathbf{Z}^n$, $\mathbf{u}$ is the new coordinate vector and Q is a **Z**-invertible matrix. Equation (3.5) can be written as

$$\mathbf{x} = u_1\mathbf{q}_1 + u_2\mathbf{q}_2 + \cdots u_n\mathbf{q}_n$$

where $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are the columns of Q. This however means that the new coordinates are taken with respect to the basis consisting of the columns of

Q. Since the change of coordinates matrix expressing the new coordinates in the terms of the old is the inverse of Q, we arrive again at the formula (3.4).

From Proposition 3.4 and Proposition 3.5 we get the following important corollary.

Corollary 3.6 *Let $f : \mathbf{Z}^n \to \mathbf{Z}^m$ be a homomorphism of groups. Let the columns of $V := [\mathbf{v}_1\mathbf{v}_2 \ldots \mathbf{v}_n]$ and $W := [\mathbf{w}_1\mathbf{w}_2 \ldots \mathbf{w}_m]$ be bases respectively in $\mathbf{Z}^n$ and $\mathbf{Z}^m$. Then the matrix B of f in these bases may be obtained from the matrix of f in the canonical bases by the formula*

$$B = W^{-1} A_f V.$$

Example 3.7 Consider a homomorphism $f : \mathbf{Z}^3 \to \mathbf{Z}^3$ given in the canonical bases by the matrix

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

It can be verified that the following matrices

$$R = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

are $\mathbf{Z}$-invertible, hence the columns of each form a basis for $\mathbf{Z}^3$ (see Exercise 3.3). Consider a new basis for $\mathbf{Z}^3$ viewed as the domain of f formed by the columns of R and a new basis for $\mathbf{Z}^3$ viewed as the target space of f formed by the columns of Q.

It is easy to verify that

$$B := Q^{-1} A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By Corollary 3.6, B is the matrix of f with respect to the new bases. Note that we obtained a diagonal matrix which is much simpler to study than the original matrix A. At present, the matrixes R and Q seem to be taken out of blue, but we will go back to this topic and show later how to obtain them.

There is another way of viewing Corollary 3.6 in the light of the previous discussion on the relation between the change of coordinate formula (3.5) and the change of basis. Suppose we have a change of coordinates in $\mathbf{Z}^n$ given by the formula

$$\mathbf{x} = R\mathbf{u}$$

and another one in $\mathbf{Z}^m$ given by

$$\mathbf{y} = Q\mathbf{v}.$$

If we substitute the old coordinates by the new ones in the equation

$$\mathbf{y} = A_f\mathbf{x},$$

we get

$$Q\mathbf{v} = A_f R\mathbf{u}$$

which is equivalent to

$$\mathbf{v} = Q^{-1}A_f R\mathbf{u}.$$

If we substitute $B := Q^{-1}A_f R$, we get the equation

$$\mathbf{v} = B\mathbf{u}.$$

Thus, in another way we come to the same conclusion that B is the matrix of f with respect to the new coordinates. As we already noticed, the columns of R and Q, are the coordinates of the new basis vectors in $\mathbf{Z}^n$ and, respectively, $\mathbf{Z}^m$.

There are three elementary ways to obtain a new basis from an old basis. If $V := \{v_1, v_2, \dots, v_n\}$ is a basis in G then the new basis may be obtained by

(b1) Exchanging elements v_i and v_j ,
(b2) Multiplying v_i by -1, [1]
(b3) Replacing v_j by $v_j + qv_i$, where $q \in \mathbf{Z}$.

The fact that operations (b1) and (b2) transform a basis into a basis is straightforward. The same fact concerning (b3) is the contents of Exercise 3.1.

Performing these operations on the canonical basis in $\mathbf{Z}^n$ one obtains the bases which are the columns of the matrices

$$E_{i,j} := \left[\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & \cdot & \cdot & 0 & 1 & & & \\
& & & \cdot & 1 & & & 0 & & & \\
& & & \cdot & & \cdot & & \cdot & & & \\
& & & 0 & & & 1 & \cdot & & & \\
& & & 1 & 0 & \cdot & \cdot & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{array}\right]
\begin{array}{c}
\\ \\ \\ (i) \\ \\ \\ \\ (j) \\ \\ \\ \\
\end{array},$$

---

[1] In a linear algebra course the analogy of (b2) is multiplication by any nonzero real number. This is because every nonzero real number is invertible, i.e. $a\frac{1}{a} = 1$, but in the integers only ± 1 have multiplicative inverses.

$$E_i := \begin{bmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ (i) \\ \\ \\ \\ \end{matrix} ,$$

and

$$E_{i,j,q} := \begin{bmatrix} 1 & & & & & & & & \\ & \cdot & & & & & & & \\ & & 1 & 0 & \cdot & 0 & q & & \\ & & & 1 & & & 0 & & \\ & & & & \cdot & & \cdot & & \\ & & & & & 1 & 0 & & \\ & & & & & & 1 & & \\ & & & & & & & \cdot & \\ & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ (i) \\ \\ \\ \\ (j) \\ \\ \\ \end{matrix} ,$$

where all empty spaces denote zero. As we will see in the sequel, these matrices are very important. They are called *elementary matrices*.

Notice that all three elementary matrices are **Z**-invertible. Indeed,

$$E_{i,j}^{-1} = E_{i,j}, \quad E_i^{-1} = E_i \quad \text{and} \quad E_{i,j,q}^{-1} = E_{i,j,-q}. \tag{3.6}$$

In particular, finding the inverse of an elementary matrix is extremly easy, unlike the case of a general integer matrix.

Let A be a fixed matrix. Denote by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ its rows and by $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ its columns. One easily verifies that the multiplication of A from the left by one of the matrices $E_{i,j}$, E_i, $E_{i,j,q}$ is equivalent to performing respectively one of the following elementary operations on rows of A.

(r1) Exchange rows $\mathbf{r}_i$ and $\mathbf{r}_j$;
(r2) Multiply $\mathbf{r}_i$ by -1 ;
(r3) Replace $\mathbf{r}_i$ by $\mathbf{r}_i + q\mathbf{r}_j$, where $q \in \mathbf{Z}$.

Similarly, the multiplication of A from the right by one of the matrices $E_{i,j}$, E_i, $E_{i,j,q}$ is equivalent to performing respectively one of the following elementary operations on columns of A.

(c1) Exchange columns $\mathbf{c}_i$ and $\mathbf{c}_j$;
(c2) Multiply $\mathbf{c}_j$ by -1 ;
(c3) Replace $\mathbf{c}_j$ by $\mathbf{c}_j + q\mathbf{c}_i$, where $q \in \mathbf{Z}$,

These are, in fact, row operations on the transposed matrix A^T.

Example 3.8 Let A be a 5×3 matrix. If we wish to exchange the second and third column, this can be done by the elementary matrix

$$E_{3,5} = \begin{bmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$$

since

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{13} & a_{14} & a_{15} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \begin{bmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & a_{12} & a_{14} & a_{15} \\ a_{21} & a_{23} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{33} & a_{32} & a_{34} & a_{35} \end{bmatrix}.$$

Notice that the matrices $E_{i,j}$, E_i, $E_{i,j,q}$ themselves may be obtained by performing respectively the operations (r1), (r2) and (r3) or (c1), (c2) and (c3) on the identity $m \times m$ matrix $I_{m\times m}$.

Since elementary row operations on a matrix $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ are equivalent to the multiplication of A on the left by elementary $m \times m$ matrices, they are related to changes of basis in $\mathbf{Z}^m$ which is the target space of A viewed as a mapping from $\mathbf{Z}^n$ to $\mathbf{Z}^m$. Similarly, elementary column operations are equivalent to the multiplication of A on the left by elementary $n \times n$ matrices, so they are related to changes of basis in $\mathbf{Z}^n$ which is the domain space of A. The relation between those matrices on the left or right of A and new bases of $\mathbf{Z}^m$ or, respectively, $\mathbf{Z}^n$, is exhibited in Corollary 3.6.

A natural question to ask is how do elementary operations affect the image and kernel of A. We start from a proposition about the column operations because we need it in the sequel, and we leave the analogous statement for row operations as an exercise.

Proposition 3.9 *Let A be an $m\times n$ integer matrix and let B be obtained from A by elementary column operations. More precisely, let $B := AR$ where R is the product of elementary $n \times n$ matrices representing the column operations. Then*

$$\operatorname{im} A = \operatorname{im} B$$

and

$$\ker A = R(\ker B).$$

Proof. Since R is invertible, it is an epimorphism so

$$\operatorname{im} B = \{B\mathbf{x} \mid \mathbf{x} \in \mathbf{Z}^m\} = \{AR\mathbf{x} \mid \mathbf{x} \in \mathbf{Z}^m\} = \{A\mathbf{y} \mid \mathbf{y} \in \mathbf{Z}^m\} = \operatorname{im} A.$$

Again, since R is invertible, for any $\mathbf{x} \in \mathbf{Z}^n$,

$$A\mathbf{x} = 0 \iff BR^{-1}\mathbf{x} = 0 \iff R^{-1}\mathbf{x} \in \ker B \iff \mathbf{x} \in R(\ker B).$$

□

Example 3.10 Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

We want to perform column operations on A. It is customary to carry out this task using row operations on the transposed matrix

$$A^T = \begin{bmatrix} 3 & 2 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

A practical way of keeping track of elementary operations is by row reducing the augmented matrix:

$$\left[I_{3\times 3} | A^T \right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \begin{array}{c} -\mathbf{r}_2 \\ \mapsto \\ \mapsto \end{array} \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

$$\begin{array}{c} \mapsto \\ -2\mathbf{r}_1 \\ \mapsto \end{array} \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & 2 \\ -2 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \begin{array}{c} \mapsto \\ +\mathbf{r}_3 \\ \mapsto \end{array} \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & 2 \\ -2 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

$$\begin{array}{c} +2\mathbf{r}_2 \\ \mapsto \\ \mapsto \end{array} \left[\begin{array}{ccc|cc} -3 & 5 & 2 & 1 & 0 \\ -2 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \begin{array}{c} \mapsto \\ \times(-1) \\ \mapsto \end{array} \left[\begin{array}{ccc|cc} -3 & 5 & 2 & 1 & 0 \\ 2 & -3 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

$$\begin{array}{c} \mapsto \\ \mapsto \\ -3\mathbf{r}_2 \end{array} \left[\begin{array}{ccc|cc} -3 & 5 & 2 & 1 & 0 \\ 2 & -3 & -1 & 0 & 1 \\ -6 & 9 & 4 & 0 & 0 \end{array}\right] := \left[P | B \right].$$

Note that P is the product of the elementary matrices corresponding to row operations, thus $B = PA^T$. Furthermore, B is a very simple matrix. In the next section we will present an algorithm performing such a simplification.

In order to see the result of column operations, consider

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$B^T = (PA^T)^T = AP^T = AR$$

where

$$R = P^T = \begin{bmatrix} -3 & 2 & -6 \\ 5 & -3 & 9 \\ 2 & -1 & 4 \end{bmatrix}$$

is the product of elementary matrices corresponding to column operations.

What information about A can we extract from the above calculations?

First, by Corollary 3.6, the columns of R form the new basis for the domain $\mathbf{Z}^3$ with respect to which the corresponding homomorphism f_A is represented by the new matrix B^T.

Next, $\operatorname{im} A$ is equal to $\operatorname{im} B^T = \mathbf{Z}^2$ by Proposition 3.9, thus A is an epimorphism. This leads to an interesting observation. Since $\operatorname{im} A = \mathbf{Z}^2$ is generated by the columns $A\mathbf{e}_1$, $A\mathbf{e}_2$, $A\mathbf{e}_3$ of A, the elements $(3,2), (2,0)$ and $(0,3)$ of $\mathbf{Z}^2$ generate the whole group $\mathbf{Z}^2$. But the reader may verify that no two of them do. Therefore a result known from linear algebra stating that from any set of generators of a vector space one can extract a basis does not generalize to free abelian groups.

Finally, we identify $\ker A$. It is easy to verify that $\ker B^T$ is generated by

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so, by Proposition 3.9, $\ker A$ is generated by the third column of R denoted by $\mathbf{r}_3$:

$$\ker A = \langle \mathbf{r}_3 \rangle = \mathbf{Z} \begin{bmatrix} -6 \\ 9 \\ 4 \end{bmatrix}.$$

The following straightforward proposition will be needed in the sequel.

Proposition 3.11 *Elementary row operations leave zero columns intact. Similarly, elementary column operations leave zero rows intact.*

Another useful feature is as follows.

Proposition 3.12 *Assume A is an integer matrix and A' results from applying a sequence of elementary row and/or column operations to A. If $p \in \mathbf{Z}$ divides all entries of A, then it divides all entries of A'.*

Proof. In the case A' results from A by applying only one elementary row or column operation the conclusion follows immediately from the form of elementary row and column operations. The general case follows by an easy induction argument. □

We finish this section with two algorithms. Some comments on notation are in order. When presenting the algorithm we will use $*$ to denote the multiplication of matrices. Also, whenever we want to store simultaneously a matrix A and its inverse, we will use $\bar{A}$ as the variable name for the inverse.

Algorithm 3.13 Elementary Row Operation

```
function rowOperation(matrix B, Q, Q̄, E)
B := E * B;
Q̄ := E * Q̄;
Q := Q * E^{-1};
return (B, Q, Q̄);
```

A careful reader may have noticed that the above algorithm is in general impractical to implement, because it requires finding the inverse of an integer matrix. However, we will only use this algorithm with argument `E` containing an elementary matrix. By (3.6) we have an explicit formula for the inverse. Actually, as we already pointed out, also the matrix multiplication in this case may be replaced by a respective operation on the rows of the second operand. Therefore, in the case when the last argument is an elementary matrix, the implementation of the algorithm is straightforward, although in general it requires rewriting it as three separate algorithms, one for each type of elementary matrix (see Exercise 3.6).

Proposition 3.14 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$. Let $R \in \mathrm{M}_{n,n}(\mathbf{Z})$, $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ be $\mathbf{Z}$-invertible matrices such that*

$$B = Q^{-1}AR.$$

If $E \in \mathrm{M}_{m,m}(\mathbf{Z})$ is a $\mathbf{Z}$-invertible matrix, then Algorithm 3.13 applied to (B, Q, Q^{-1}, E) returns a matrix $B' \in \mathrm{M}_{m,n}(\mathbf{Z})$ and mutually inverse $\mathbf{Z}$-invertible matrices $Q', \bar{Q}' \in \mathrm{M}_{m,m}(\mathbf{Z})$ such that

$$B' = (Q')^{-1}AR = \bar{Q}'AR.$$

Moreover, if E is a matrix of an elementary row operation, then B' is the outcome of applying this operation to B.

Proof. We have

$$Q' = QE^{-1}$$

and

$$\bar{Q}' = E\bar{Q}.$$

Therefore

$$Q'\bar{Q}' = QE^{-1}E\bar{Q} = I_{m\times m}$$

and

$$\bar{Q}'Q' = E\bar{Q}QE^{-1} = I_{m\times m}$$

which proves that Q' and $\bar{Q}'$ are mutually inverse $\mathbf{Z}$-invertible matrices. Moreover,

$$B' = EB = EQ^{-1}AR = E\bar{Q}AR = \bar{Q}'AR = (Q')^{-1}AR.$$

The remaining assertion is straightforward. □

Similarly we construct the algorithm performing column operations.

Algorithm 3.15 Elementary Column Operation

```
function columnOperation(matrix B, R, R̄, E)
B := B * E;
R̄ := E^{-1} * R̄;
R := R * E;
return (B, R, R̄);
```

As discussed earlier, notice that in the case when the last argument is an elementary matrix, the algorithm may be rewritten in such a way that neither matrix multiplication nor matrix inverse is needed.

Proposition 3.16 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$. Let $R \in \mathrm{M}_{n,n}(\mathbf{Z})$, $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ be $\mathbf{Z}$-invertible matrices such that*

$$B = Q^{-1}AR.$$

If $E \in \mathrm{M}_{m,m}(\mathbf{Z})$ is a $\mathbf{Z}$-invertible matrix, then Algorithm 3.13 applied to (B, R, R^{-1}, E) returns a matrix $B' \in \mathrm{M}_{m,n}(\mathbf{Z})$ and mutually inverse $\mathbf{Z}$-invertible matrices $R', \bar{R}' \in \mathrm{M}_{m,m}(\mathbf{Z})$ such that

$$B' = Q^{-1}AR'.$$

Moreover, if E is a matrix of an elementary row operation, then B' is the outcome of applying this operation to B.

Exercises

3.1 Let $V := \{v_1, v_2, \ldots, v_n\}$ be a basis in a free abelian group G. Put $V' := \{v'_1, v'_2, \ldots, v'_n\}$, where

$$v'_k = \begin{cases} v_k \text{ if } k \neq j \\ v_j + qv_i \text{ otherwise.} \end{cases}$$

for some $i, j \in \{1, 2, \ldots n\}$ and $q \in \mathbf{Z}$. Show that V' is also a basis in G.

3.2 Let $f : \mathbf{Z}^n \to \mathbf{Z}^n$ be an isomorphism. Prove that $A_{f^{-1}} = A_f^{-1}$.

3.3 Let A be a $\mathbf{Z}$-invertible $n \times n$ matrix. Show that the columns of A form a basis of $\mathbf{Z}^n$.

3.4 * Let $V = \{v_1, v_2, \ldots v_n\}$ and $V' = \{v'_1, v'_2, \ldots v'_n\}$ be two bases of a free abelian group G. This implies that every $v'_i \in V'$ can be written as a linear combination

$$v'_i = \sum_{j=1}^{n} p_{ij} v_j.$$

Let $P := [p_{ij}]$. Show that the inverse change of coordinates matrix is related to P by the formula

$$A_{\xi_V \xi_{V'}^{-1}} = P^T.$$

3.5 Let A be an $m \times n$ integer matrix and let B be obtained from A by elementary row operations. More precisely, let $B := PA$ where P is the product of elementary $m \times m$ matrices representing the row operations. Show that

$$\operatorname{im} B = \operatorname{im} PA$$

and that

$$\ker B = \ker A.$$

3.6 Rewrite Algorithm 3.15 as three algorithms, one for each type of elementary matrix, in such a way that it does not require either matrix multiplication or finding the matrix inverse but only the elementary operations on rows. Do the same with Algorithm 3.13.

3.2 Row Echelon Form

We will use the elementary row and column operations from the previous section to transform arbitrary integer matrices into simpler matrices. The steps we present should remind the reader of Gaussian elimination, with the constraint that we must always use integer coefficients. Our eventual goal is to produce "diagonal" matrices, since as in Example 3.10, in this case the image and kernel of the map are immediately evident. We put the word diagonal in quotation marks to emphasize that we are dealing with $m \times n$ matrices. As a first step towards diagonalization, we introduce the following notion which generalizes the concept of an upper triangular matrix.

Definition 3.17 The *pivot position* of a non-zero vector is the position of the first non-zero element of this vector. A matrix A is in *row echelon form* if for any two consecutive rows $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$, if $\mathbf{r}_{i+1} \neq 0$ then $\mathbf{r}_i \neq 0$ and the pivot position of $\mathbf{r}_{i+1}$ is greater than the pivot position of $\mathbf{r}_i$.

Notice that if a matrix is in row echelon form then all its non-zero rows come first, followed by zero rows, if there are any.

Example 3.18 The following matrix is in row echelon form

$$B = \begin{bmatrix} 0 & 2 & 0 & 7 & 3 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows are the non-zero rows. The pivot positions of these three rows are respectively $2, 3$ and 5.

The following proposition is straightforward.

Proposition 3.19 *Assume $A = [a_{ij}]$ is an $m \times n$ matrix in row echelon form and the non-zero rows are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$. If l_i is the pivot position of $\mathbf{r}_i$ then $\{l_i\}_{i=1}^k$ is a strictly increasing sequence and $a_{i'l_i} = 0$ for $i' > i$.*

A matrix A is in *column echelon form* if its transpose A^T is in row echelon form. For our purposes, column echelon form is more useful than row echelon form but, as it was observed in Example 3.10, row operations and row echelon form are more customary than the column ones. Therefore we will often turn problems concerning a matrix A around by considering its transpose A^T, bringing it to row echelon form and transposing the resulting matrix back.

As the following proposition indicates if a matrix is in column echelon form, then one has an immediate description of the image and the kernel of the matrix.

Proposition 3.20 *Suppose that $A = [a_{ij}] \in \mathrm{M}_{m,n}(\mathbf{Z})$ is in column echelon form. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the columns of A and $\mathbf{v}_k$ is the last non-zero column, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis of* $\operatorname{im} A$ *and $\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots \mathbf{e}_n\}$ is a basis of* $\ker A$.

Proof. By definition, A^T is in row echelon form. The rows of A^T are $\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots \mathbf{v}_n^T$ and since A^T is in row echelon form, the non-zero rows of A^T are $\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots \mathbf{v}_k^T$ for some $k \leq n$. Let l_j be the first non-zero element in row $\mathbf{v}_j^T$. Then, by Proposition 3.19 $\{l_j\}_{j=1}^k$ is a strictly increasing sequence. We will show that $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly independent. Let $\sum_{j=1}^k \alpha_j \mathbf{v}_j = 0$. Let $\pi_i : \mathbf{Z}^m \to \mathbf{Z}$ denote the projection onto the i-th coordinate. Then for $s = 1, 2, \ldots k$

$$0 = \pi_{l_s}(\sum_{j=1}^k \alpha_j \mathbf{v}_j) = \sum_{j=1}^k \alpha_j \pi_{l_s}(\mathbf{v}_j) = \sum_{j=1}^k \alpha_j a_{l_s j} = \sum_{j=1}^s \alpha_j a_{l_s j}, \tag{3.7}$$

because by Proposition 3.19 $a_{l_s j} = a^T_{j l_s} = 0$ for $j > s$. In particular $0 = \alpha_1 a_{l_1 1}$ and since $a_{l_1 1} \neq 0$, we get $\alpha_1 = 0$. Assuming that $\alpha_1 = \alpha_2 = \ldots \alpha_{s-1} = 0$ for some $s \leq k$, we get from (3.7) that $0 = \alpha_s a_{l_s s}$ and consequently $\alpha_s = 0$. This proves that $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly independent. Obviously they generate $\operatorname{im} A$. Therefore they constitute a basis of $\operatorname{im} A$.

To show that $\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots \mathbf{e}_n\}$ constitute a basis of $\ker A$ take

$$\mathbf{x} = (x_1, x_2, \ldots x_n) = \sum_{j=1}^n x_j \mathbf{e}_j \in \ker A.$$

Then

$$0 = A\mathbf{x} = \sum_{j=1}^n x_j A\mathbf{e}_j = \sum_{j=1}^n x_j \mathbf{v}_j = \sum_{j=1}^k x_j \mathbf{v}_j.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly independent, $x_1 = x_2 = \ldots x_k = 0$. Therefore $\mathbf{x} = \sum_{j=k+1}^n x_j \mathbf{e}_j$, which shows that $\{\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots \mathbf{e}_n\}$ generate $\ker A$. Since obviously they are linearly independent, they form a basis of $\ker A$. □

Proposition 3.20 provides justification for the traditional terminology of referring to the group $\operatorname{im} A$ as the *column space of* A.

Let `A` be a variable of type **matrix** storing an $m \times n$ matrix A with entries a_{ij}, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. In what follows we will need to be able to refer to submatrices of A consisting of all entries $a_{i,j}$ such that $i \in \{k, \ldots, l\}$ and $j \in \{k', \ldots, l'\}$ for some $k \leq l$ and $k' \leq l'$. In the algorithms these submatrices will be extracted from the variable `A` by writing

$\mathtt{A}[k:l,k':l']$ (see Section 14.2.4). To be consistent, in the discussion of the algorithms we will denote this submatrix by $A[k:l,k':l']$. In keeping with this notation we will often write $A[i,j]$ to refer to the entry $a_{i,j}$. In order to avoid separate analysis of special cases it will be convenient to use the notation for submatrices also when $k>l$ or $k'>l'$. In this case we will treat $A[k:l,k':l']$ as a zero matrix.

Our goal in this section is to develop an algorithm which brings an integer matrix $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ to row echelon form by means of elementary row operations. We will achieve this goal gradually by working on the matrix row by row. Therefore we will need the following concept. We say that a matrix A satisfies the (k,l) *criterion of row echelon form* if the submatrix $A[1:m,1:l]$ consisting of the first l columns of A is in row echelon form and the non zero rows of this submatrix are exactly the first k rows.

Example 3.21 The following matrix

$$\begin{bmatrix} 2&0&1&0&1&0\\ 0&0&1&4&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&3&0\\ 0&0&0&0&0&2 \end{bmatrix}$$

is not in row echelon form, but it satisfies the $(2,3)$ criterion of row echelon form. It also satisfies the $(2,4)$ criterion of row echelon form, but satisfies neither the $(2,2)$ nor the $(2,5)$ criterion of row echelon form.

Assume $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ and $k \in \{0,1,2,\ldots m\}$, $l \in \{0,1,2,\ldots n\}$. The following four propositions will turn out useful in induction arguments based on the (k,l) criterion of row echelon form.

Proposition 3.22 *Every matrix satisfies the* $(0,0)$ *criterion of row echelon form. If* A *satisfies the* (k,n) *criterion of row echelon form, then* A *is in row echelon form and the non zero rows of* A *are exactly the first* k *rows.* □

Proposition 3.23 *Assume* A *satisfies the* $(k-1,l-1)$ *criterion of row echelon form,* $A[k,l]\neq 0$ *and* $A[k+1:m,l]=0$*. Then* A *satisfies the* (k,l) *criterion of row echelon form.*

Proof. The pivot position of the $(k-1)$-st row of A is at most $l-1$ and the pivot position of the k-th row is exactly l. The last $m-k$ rows of $A[1:m,1:l]$ are zero. The conclusion follows. □

Proposition 3.24 *Assume* A *satisfies the* (k,l) *criterion of row echelon form. If* $A[k+1:m,l+1:l']=0$ *for some* $l' \in \{0,1,2,\ldots n\}$*, then* A *satisfies the* (k,l') *criterion of row echelon form.*

Proof. The pivot positions of the first k rows of $A[1:k,1:l']$ and $A[1:k,1:l]$ are the same. The last $m-k$ rows of $A[1:k,1:l']$ are zero. The conclusion follows. □

Proposition 3.25 *Assume A satisfies the (k,l) criterion of row echelon form and A' results from applying an elementary row operation which does not change the first k rows. Then the first k rows and the first l columns of A and A' coincide.*

Proof. Obviously the first k rows of A cannot change. The first l columns can change only in the entries of the submatrix $A[k+1:m, 1:l]$ but this part of A is zero by the assumption that A satisfies the (k,l) criterion of row echelon form. □

The operation which is fundamental to most of the algorithms presented in this chapter is that of row reduction. It consists in applying suitable elementary row operations in order to achieve the situation that all elements in a given column below a selected position are zero. In the case of matrices with real entries this is a simple procedure involving division. Unfortunately, dividing one integer by another does not necessarily produce an integer. This, of course, creates a problem in our setting which we need to circumvent.

To begin with consider the case of an integer matrix $B \in \mathrm{M}_{m,n}(\mathbf{Z})$ satisfying the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2. \ldots m\}$, $l \in \{1, 2, \ldots n\}$ and such that $B[k,l] \neq 0$. Our modest goal is to decrease the magnitudes of the entries in the subcolumn $B[k+1:m, l]$.

Algorithm 3.26 Partial Row Reduction

```
function partRowReduce(matrix B, Q, Q̄, int k, l)
for i := k + 1 to numberOfRows(B) do
    q := floor(B[i, l]/B[k, l]);
    (B, Q, Q̄) := rowOperation(B, Q, Q̄, E_{i,k,-q});
endfor;
return (B, Q, Q̄);
```

Example 3.27 Let

$$A = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 3 & 2 & 1 & 4 \\ 4 & 4 & -2 & -2 \end{bmatrix}$$

and let $(B, Q, \bar{Q}) = \mathtt{partRowReduce}(A, I_{m\times m}, I_{m\times m}, 1, 1)$. Then

$$B = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 1 & -1 & 0 & 5 \\ 0 & -2 & -4 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Before stating the properties of Algorithm 3.26 we introduce the following notation concerning the subcolumn $B[k:m, l]$ of a matrix $B \in \mathrm{M}_{m,n}(\mathbf{Z})$

$$\alpha_{kl}(B) := \begin{cases} \min\{|B[i,l]| \mid i \in [k,m], B[i,l] \neq 0\} & \text{if } B[k:m, l] \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this chapter this quantity will be useful for measuring the progress in decreasing the magnitudes of the entries in the subcolumns of B.

Proposition 3.28 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \ldots m\}$, $l \in \{1, 2, \ldots n\}$ and*

$$B = Q^{-1}AR$$

for some $\mathbf{Z}$-invertible matrices $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ and $R \in \mathrm{M}_{n,n}(\mathbf{Z})$. If $B[k,l] = \alpha_{kl}(B) \neq 0$, then Algorithm 3.26 applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in \mathrm{M}_{m,n}(\mathbf{Z})$ and mutually inverse $\mathbf{Z}$-invertible matrices $Q', \bar{Q}' \in \mathrm{M}_{m,m}(\mathbf{Z})$ such that:

1. *the first k rows and $l-1$ columns of B and B' coincide,*
2. *$\alpha_{kl}(B') < \alpha_{kl}(B)$ or $B'[k+1 : m, l] = 0$,*
3. *$B' = (Q')^{-1}AR = \bar{Q}'AR$.*

Proof. The first property follows from Proposition 3.25. The outcome of the substitution $\mathtt{B} := E_{\mathtt{i},\mathtt{l},-\mathtt{q}} * \mathtt{B}$ for $i = k+1, k+2, \ldots m$ is that $B'[i,l]$ is the remainder from dividing $B[i,l]$ by $B[k,l]$. If at least one of the remainders is non-zero, then $\alpha_{kl}(B')$ is the smallest non zero remainder which obviously is less than $B[k,l] = \alpha_{kl}(B)$. Otherwise all remainders are zero, i.e. $B[k+1 : m, l] = 0$. This proves the second property. The final property follows from Proposition 3.14 by a straightforward induction argument. □

Of course Algorithm 3.26 works as desired only when $B[k,l] = \alpha_{kl}(B) \neq 0$, which obviously is a very special case. Thus, the next step is to develop an algorithm that brings an integer matrix into this special form. We begin by a very simple procedure that (assuming it exists) identifies the first entry with the smallest nonzero magnitude in a part of a vector.

Algorithm 3.29 Smallest Nonzero Entry

```
function smallestNonzero(vector v, int k)
alpha := min {abs(v[i]) | i in [k : length(v)] and v[i] ≠ 0} ;
i0 := min {i | i in [k : length(v)] and abs(v[i]) = alpha} ;
return (alpha, i0);
```

We make use of Algorithm 3.29 as follows.

Algorithm 3.30 Row Preparation

```
function rowPrepare(matrix B, Q, Q̄, int k, l)
m := numberOfRows(B);
(α, i) := smallestNonzero(B[1 : m, l], k);
(B, Q, Q̄) := rowOperation(B, Q, Q̄, E_{k,i});
return (B, Q, Q̄);
```

Example 3.31 Let

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & -1 \\ 4 & 4 & -2 & -2 \end{bmatrix}.$$

Let $(B, Q, \bar{Q}) = \texttt{rowPrepare}(A, I_{m\times m}, I_{m\times m}, 1, 1)$. Then,

$$B = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 3 & 2 & 1 & 4 \\ 4 & 4 & -2 & -2 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \bar{Q} = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proposition 3.32 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \dots m\}$, $l \in \{1, 2, \dots n\}$ and*

$$B = Q^{-1}AR$$

for some $\mathbf{Z}$-invertible matrices $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ and $R \in \mathrm{M}_{n,n}(\mathbf{Z})$. If $B[k : m, l] \neq 0$ then Algorithm 3.30 applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in \mathrm{M}_{m,n}(\mathbf{Z})$ and mutually inverse $\mathbf{Z}$-invertible matrices $Q', \bar{Q}' \in \mathrm{M}_{m,m}(\mathbf{Z})$ such that:

1. *the first $k-1$ rows and $l-1$ columns of B and B' coincide,*
2. *$B'[k, l] = \alpha_{kl}(B') = \alpha_{kl}(B) \neq 0$,*
3. *$B' = (Q')^{-1}AR = \bar{Q}'AR$.*

Proof. The first property follows from Proposition 3.25. The remaining properties are straightforward. □

We can combine these algorithms to row reduce the first column of an integer matrix.

Algorithm 3.33 Row Reduction

```
function rowReduce(matrix B, Q, Q̄, int k, l)
m := numberOfRows(B);
while B[k + 1 : m, l] ≠ 0 do
    (B, Q, Q̄) := rowPrepare(B, Q, Q̄, k, l);
    (B, Q, Q̄) := partRowReduce(B, Q, Q̄, k, l);
endwhile ;
return (B, Q, Q̄);
```

Proposition 3.34 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \dots m\}$, $l \in \{1, 2, \dots n\}$ and*

$$B = Q^{-1}AR$$

for some $\mathbf{Z}$-invertible matrices $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ and $R \in \mathrm{M}_{n,n}(\mathbf{Z})$. If $B[k : m, l] \neq 0$ then Algorithm 3.33 applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in \mathrm{M}_{m,n}(\mathbf{Z})$ and mutually inverse $\mathbf{Z}$-invertible matrices $Q', \bar{Q}' \in \mathrm{M}_{m,m}(\mathbf{Z})$ such that:

1. *$B' = (Q')^{-1}AR = \bar{Q}'AR$,*
2. *B' satisfies the (k, l) criterion of row echelon form.*

Proof. Let $B^{(i)}$ denote the value of variable B after completing the i-th pass of the **while** loop. First observe that on every call of `rowPrepare` the assumptions of Proposition 3.32 are satisfied and by this proposition on every call of `partRowReduce` the assumptions of Proposition 3.28 hold. By Proposition 3.32 `rowPrepare` does not change the value of α_{kl} and by Proposition 3.28 `partRowReduce` decreases it unless $B^{(i)}[k+1:m,l]=0$, which can happen only in the last pass through the **while** loop. Therefore the sequence

$$\alpha_i := \alpha_{kl}(B^{(i)})$$

is a decreasing sequence of positive integers. Hence it must be finite and the **while** loop must be completed. It follows that the algorithm always halts and $B' = B^{(i_*)}$, where i_* denotes the number of iterations through the **while** loop. The only way to leave the **while** loop is when the test fails, which implies

$$B'[k+1:m,l]=0. \tag{3.8}$$

Since by Proposition 3.28 and Proposition 3.32 the first $l-1$ columns of $B^{(i)}$ do not depend on i, B' satisfies the $(k-1,l-1)$ criterion of row echelon form. Proposition 3.23, property (3.8) and Proposition 3.32.2 imply the second property.

The first property follows as an immediate consequence of the corresponding properties of the `rowPrepare` and `partRowReduce` algorithms. □

Example 3.35 Let

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & -1 \\ 4 & 4 & -2 & -2 \end{bmatrix}.$$

After the first iteration of the while statement in Algorithm 3.33 called with arguments $(A, I_{m\times m}, I_{m\times m}, 1, 1)$

$$\mathtt{B} = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 1 & -1 & 0 & 5 \\ 0 & -2 & -4 & 0 \end{bmatrix} \quad \mathtt{Q} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \bar{\mathtt{Q}} = \mathtt{Q}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Thus, after another application of `rowPrepare`

$$\mathtt{B} = \begin{bmatrix} 1 & -1 & 0 & 5 \\ 2 & 3 & 1 & -1 \\ 0 & -2 & -4 & 0 \end{bmatrix} \quad \mathtt{Q} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \bar{\mathtt{Q}} = \mathtt{Q}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

After the application of `partRowReduce`

$$\mathtt{B} = \begin{bmatrix} 1 & -1 & 0 & 5 \\ 0 & 5 & 1 & -11 \\ 0 & -2 & -4 & 0 \end{bmatrix} \quad \mathtt{Q} = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad \bar{\mathtt{Q}} = \mathtt{Q}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

With regard to Example 3.35, observe that we have now reduced the problem of finding a row-echelon reduction of A, to the problem of finding a row-echelon reduction of

$$B[2:3,2:4] = \begin{bmatrix} 5 & 1 & -11 \\ -2 & -4 & 0 \end{bmatrix}$$

This motivates the following algorithm.

Algorithm 3.36 Row Echelon

```
function rowEchelon(matrix B)
m := numberOfRows(B);
n := numberOfColumns(B);
Q := Q̄ := identityMatrix(m);
k := l := 0;
repeat
    while l ≤ n and B[k + 1 : m, l] = 0 do l := l + 1;
    if l = n + 1 then break endif;
    k := k + 1;
    (B, Q, Q̄) := rowReduce(B, Q, Q̄, k, l);
until k = m;
return (B, Q, Q̄, k);
```

Theorem 3.37 *Algorithm 3.36 always stops. Given a matrix $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ on input it returns a matrix $B \in \mathrm{M}_{m,n}(\mathbf{Z})$, mutually inverse $\mathbf{Z}$-invertible matrices $Q, \bar{Q} \in \mathrm{M}_{m,m}(\mathbf{Z})$ and a number k such that B is in row echelon form, exactly the first k rows of B are non zero and*

$$B = Q^{-1}A = \bar{Q}A. \tag{3.9}$$

Proof. To prove that the algorithm always halts, we need to show that every loop of the algorithm is left. Clearly the **repeat** loop is called at most m times. The **while** loop is always left, because the `l` variable is increased at every pass through this loop, so even if the condition $\mathtt{B}[\mathtt{k}+1:\mathtt{m},\mathtt{l}] = 0$ is not met, the variable `l` finally reaches $\mathtt{n}+1$.

Let $B^{(0)} := A$, $Q^{(0)} := \bar{Q}^{(0)} := I_{m\times m}$ and $l^{(0)} := 0$ denote the initial values of variables `B`, `Q`, `Q̄` and `l`. For $i = 1, 2, \ldots k$ let $B^{(i)}$, $Q^{(i)}$, $\bar{Q}^{(i)}$ and $l^{(i)}$ denote respectively the values of variables `B`, `Q`, `Q̄` and `l` on completing the i-th pass of the **repeat** loop. In particular $B^{(k)} = B$, $Q^{(k)} = Q$ and $\bar{Q}^{(k)} = \bar{Q}$.

Observe that $l^{(i)}$ is set when the **while** loop is completed on the (i)-th pass of the **repeat** loop. At this moment the `k` variable contains $i-1$ and the `B` variable contains $B^{(i-1)}$. Therefore the test in the **while** loop implies that for every $i = 1, 2, \ldots k$ we have

$$l^{(i)} > l^{(i-1)} \quad \text{implies} \quad B^{(i-1)}[i:m, l^{(i-1)} : l^{(i)} - 1] = 0 \tag{3.10}$$

and

$$B^{(i-1)}[i:m, l^{(i)}] \neq 0. \tag{3.11}$$

We will show by induction in i that for $i = 0, 1, 2, \ldots k$

$$B^{(i)} = (Q^{(i)})^{-1}A = \bar{Q}^{(i)}A, \tag{3.12}$$

$$B^{(i)} \text{ satisfies the } (i, l^{(i)}) \text{ criterion of row echelon form.} \tag{3.13}$$

Indeed, for $i = 0$ property (3.12) follows from the fact that $B^{(0)} = A$, $Q^{(0)} = \bar{Q}^{(0)} = I_{m\times m}$ and property (3.13) is vacuously fulfilled. Therefore assume that properties (3.12) and (3.13) are satisfied for some $i-1 < k$. Then $B^{(i-1)}$ satisfies the $(i-1, l^{(i-1)})$ criterion of row echelon form. However, by (3.10) and Proposition 3.24 $B^{(i-1)}$ also satisfies the $(i-1, l^{(i)}-1)$ criterion of row echelon form. By (3.11) we may apply Proposition 3.34 to $B = B^{(i-1)}$, $Q = Q^{(i-1)}$, $k = i-1$ and $l = l^{(i)} - 1$, from which we get (3.12) and (3.13).

Since $B = B^{(k)}$, $Q^{(k)} = Q$ and $\bar{Q}^{(k)} = \bar{Q}$, it follows from (3.12) that (3.9) is satisfied.

If the **repeat** loop is left when the `k` variable reaches m then all rows of B are non zero. Otherwise it is left via the **break** statement. In both cases

$$B[k+1:m, l^{(k)}:n] = 0.$$

Hence by Proposition 3.24 B satisfies the (k, n) criterion of row echelon form and by Proposition 3.22 that B is in row echelon form and exactly the first k rows of B are non zero. □

Corollary 3.38 *Every matrix with integer coefficients can be brought to a row echelon form by means of elementary row operations over* $\mathbf{Z}$.

As was pointed out in Example 3.10, the row echelon form of a matrix may be used to construct a basis of the kernel and image of that matrix.

Algorithm 3.39 Kernel-image algorithm.

```
function kernelImage(matrix B)
  m := numberOfRows(B);
  n := numberOfColumns(B);
  BT := transpose(B);
  (B, P, P̄, k) := rowEchelon(BT);
  BT := transpose(B);
  PT := transpose(P);
  return (PT[1 : m, k + 1 : n], BT[1 : m, 1 : k]);
```

Theorem 3.40 *Given an* $m \times n$ *matrix* A *on input Algorithm 3.39 returns an* $m \times k$ *matrix* V *and an* $n \times (n-k)$ *matrix* W *such that the columns of* V *constitute a basis of* $\operatorname{im} A$ *and the columns of* W *constitute a basis of* $\ker A$.

Proof. By Theorem 3.37, $B = PA^T$, B is in row echelon form and the first k rows of B are its non-zero rows. By Proposition 3.20 the first k columns of B^T constitute the basis of $\operatorname{im} B^T$ and the vectors $\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \ldots \mathbf{e}_n \in \mathbf{Z}^n$ constitute a basis of $\ker B^T$. Since $B^T = AP^T$ and P^T is invertible, Proposition 3.9 implies that $\operatorname{im} B^T = \operatorname{im} A$. Therefore the columns of V constitute a basis of $\operatorname{im} A$. Again by Proposition 3.9, $\ker A = P^T(\ker B^T)$. Therefore $P^T\mathbf{e}_{k+1}, P^T\mathbf{e}_{k+2}, \ldots, P^T\mathbf{e}_n \in \mathbf{Z}^n$ is a basis of $\ker A$. But these are exactly the columns of W. $\square$

Example 3.41 Let $A \in \mathrm{M}_{4,3}(\mathbf{Z})$ be given by

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 1 \\ 5 & 3 & -2 \end{bmatrix}.$$

We will find bases for $\ker A$ and $\operatorname{im} A$. Applying the first command of Algorithm 3.39 to A^T we get

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 4 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $k = 2$. Therefore the first two columns of B^T, i.e. $[2, 0, 4, 3]^T$ and $[0, 1, 3, 5]^T$ form a basis for $\operatorname{im} B = \operatorname{im} A$, whereas the third column of P^T, i.e. $[1, -1, 1]^T$ is a basis of $\ker A$.

Exercises

3.7 Prove Proposition 3.19.

3.8 Show that $l^{(i)}$, the value of the `l` variable on i-th pass of the **repeat** loop in Algorithm 3.36 equals the position of the i-th row of the matrix B returned by this algorithm.

3.9 Use the algorithm rowEchelon to obtain an algorithm columnEchelon computing column echelon form of a matrix.

3.3 Smith Normal Form

As was indicated earlier, the action of a linear map is most transparent when it is presented in terms of bases that result in a diagonal matrix. With this in mind we present an algorithm that produces a diagonal matrix with the property that the i-th diagonal entry divides the $(i+1)$-st diagonal entry. This latter property will be used to prove the classification theorem for finitely generated abelian groups. It should also be emphasized that the approach presented here does not lead to the most efficient algorithm. Instead we focus on expressing the essential ideas.

Since the algorithm is quite complex, we start with an example.

Example 3.42 Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

In Example 3.10 we kept track of row operations by performing them on the matrix augmented by the identity matrix on the left. Now we need to keep track of both row and column operations so we shall work with the following augmented matrix

$$\left[\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right].$$

When row operations are performed on A, only the upper blocks change. When column operations are performed on A, only the right-hand side blocks change. The zero matrix in the lower left corner never changes, it is there only to complete the expression to a square matrix. At the final stage we obtain a matrix

$$\left[\begin{array}{c|c} P & B \\ \hline 0 & R \end{array}\right]$$

where B is in the above mentioned diagonal form, R is a matrix of column operations, and P is a matrix of row operations. So, in particular we $B = PAR$.

The first step is to identify a nonzero entry of A with the minimal absolute value (we will use a bold character for it) and bring it by row and column operations to the upper left-hand corner of the matrix B obtained from A. By exchanging columns $\mathbf{c}_4$ and $\mathbf{c}_5$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & \mathbf{2} & 3 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \mathbf{2} & 3 & 3 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the first pivot entry containing 2 divides the entries in its column below it but it does not divide the entries in its line on the right of it. By subtracting $\mathbf{r}_1$ from $\mathbf{r}_2$ and $\mathbf{r}_3$, we get zero entries below the pivot 2 and then by subtracting $\mathbf{c}_4$ from $\mathbf{c}_5$ and $\mathbf{c}_6$ we reduce the value of the entries on the right of 2.

$$\mapsto \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 3 & 3 \\ -1 & 1 & 0 & 0 & -3 & -3 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & \mathbf{1} & 1 \\ -1 & 1 & 0 & 0 & -3 & -3 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now the minimal absolute value of nonzero entries of the matrix B is 1, so we repeat the procedure. By exchanging $\mathbf{c}_4$ and $\mathbf{c}_5$ we bring 1 to the upper left corner and then use a series of row and column subtractions to zero out the entries below and on the right of the pivot 1.

$$\mapsto \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \mathbf{1} & 2 & 1 \\ -1 & 1 & 0 & -3 & 0 & -3 \\ -1 & 0 & 1 & -1 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \cdots \mapsto \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 2 & 1 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now the row and column of the pivot entry 1 are in the desired form so it remains to continue reductions in the 2×2 matrix

$$B[2:3, 2:3] = \begin{bmatrix} 6 & 0 \\ \mathbf{2} & 0 \end{bmatrix}.$$

The minimal nonzero entry of that matrix is 2. By exchanging rows $\mathbf{r}_2$ with $\mathbf{r}_3$ in the augmented matrix, we bring 2 to the upper right corner of $B[2:3, 2:3]$ and by subtracting $3\mathbf{r}_2$ from $\mathbf{r}_3$ we zero out the entry 6. Thus we obtain

$$\left[\begin{array}{c|c} P & B \\ \hline 0 & R \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore the final matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is diagonal with $b_1 = 1$ dividing $b_2 = 2$. Finally, we want to know what are the changes of basis corresponding to the row and column operations. We obtained

$$R = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

In order to apply Corollary 3.6, we need

$$Q = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $B = Q^{-1}AR$, the columns of R form the new basis of $\mathbf{Z}^3$ viewed as the domain of B, expressed in terms of the canonical basis, and the columns of Q correspond to the new basis of $\mathbf{Z}^3$ viewed as the target space of B.

With the above example in mind, we now proceed with the formal description of the algorithm. Given an $m \times n$ matrix $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ our ultimate aim is to produce bases given by columns of some matrices $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ and $R \in \mathrm{M}_{n,n}(\mathbf{Z})$ such that the matrix of the homomorphism f_A in the new bases, i.e.

$$B := Q^{-1}AR$$

satisfies $B[i,j] = 0$ if $i \neq j$ and $B[i,i]$ divides $B[i+1,i+1]$. We shall do this in a recursive manner. We say that $B \in \mathrm{M}_{m,n}(\mathbf{Z})$ is in Smith form up to the k-th entry if the following three conditions are satisfied

$$B = \left[\begin{array}{ccc|c} B[1,1] & & 0 & \\ & \ddots & & 0 \\ 0 & & B[k,k] & \\ \hline & 0 & & B[k+1:m,k+1:n] \end{array}\right]$$

$$B[i,i] \text{ divides } B[i+1,i+1] \text{ for } i = 1,2,\ldots k-1 \tag{3.14}$$

and

$$B[k,k] \text{ divides } B[i,j] \quad \text{for all } i,j > k. \tag{3.15}$$

For the moment let us ignore the conditions on divisibility. Then the problem is essentially the same as that solved by `rowReduce`, except that it needs to be solved not only for the k-th column, but simultaneously for the k-th column and the k-th row.

First we extend `smallestNonzero` to find an entry which has the smallest nonzero magnitude for the entire submatrix $B[k:m,k:n]$.

Algorithm 3.43 Minimal Nonzero Entry

```
function minNonzero(matrix B, int k)
vector v, q;
for i := 1 to numberOfRows(B)
    if i < k then
        v[i] := q[i] := 0
    else
        (v[i], q[i]) := smallestNonzero(B[i, 1 : numberOfColumns(B)], k)
    endif;
endfor;
(alpha, i0) := smallestNonzero(v, k);
return (alpha, i0, q(i0));
```

Having found the entry with minimal nonzero magnitude we now need to move it to the (k, k) position. We leave it to the reader to check that this is done by the following algorithm.

Algorithm 3.44 Move Minimal Nonzero Entry

```
function moveMinNonzero(matrix B, Q, Q̄, R, R̄, int k)
(alpha, i, j) := minNonzero(B, k);
    (B, Q, Q̄) := rowOperation(B, Q, Q̄, E_{k,i});
    (B, R, R̄) := columnOperation(B, R, R̄, E_{k,j});
return (B, Q, Q̄, R, R̄);
```

We now turn to the consideration of condition (3.15). The following algorithm checks if the (k, k) entry of a matrix B divides all entries in the submatrix $B[k + 1 : m, k + 1 : n]$.

Algorithm 3.45 Check for Divisibility.

```
function checkForDivisibility(matrix B, int k)
for i := k + 1 to numberOfRows(B)
    for j := k + 1 to numberOfColumns(B)
        q := floor(B[i, j]/B[k, k]);
        if q * B[k, k] ≠ B[i, j] then
            return (false, i, j, q);
        endif;
    endfor;
endfor;
return (true, 0, 0, 0);
```

The first entry returned by this algorithm is `true` when the divisibility test succeeds. Otherwise `false` is returned together with the coordinates of the first entry where the division test fails, followed by the integer part of the quotient, which indicated failure.

Combining these algorithms we obtain an algorithm that performs one step of the recursive procedure.

Algorithm 3.46 Partial Smith Form algorithm.

function $\texttt{partSmithForm}(\textbf{matrix}\ \texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, \texttt{R}, \bar{\texttt{R}},\ \textbf{int}\ \texttt{k})$
$\texttt{m} := \textbf{numberOfRows}(\texttt{B});$
$\texttt{n} := \textbf{numberOfColumns}(\texttt{B});$
repeat
  $(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, \texttt{R}, \bar{\texttt{R}}) := \texttt{moveMinNonzero}(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, \texttt{R}, \bar{\texttt{R}}, \texttt{k});$
  $(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}) := \texttt{partRowReduce}(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, \texttt{k}, \texttt{k});$
  if $\texttt{B}[\texttt{k}+1 : \texttt{m}, \texttt{k}] \neq 0$ **then next**; **endif**;
  $(\texttt{B}, \texttt{R}, \bar{\texttt{R}}) := \texttt{partColumnReduce}(\texttt{B}, \texttt{R}, \bar{\texttt{R}}, \texttt{k}, \texttt{k});$
  if $\texttt{B}[\texttt{k}, \texttt{k}+1 : \texttt{n}] \neq 0$ **then next**; **endif**;
  $(\texttt{divisible}, \texttt{i}, \texttt{j}, \texttt{q}) := \texttt{checkForDivisibility}(\texttt{B}, \texttt{k});$
  if not `divisible` **then**
    $(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}) := \texttt{rowOperation}(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, E_{\texttt{i},\texttt{k},1});$
    $(\texttt{B}, \texttt{R}, \bar{\texttt{R}}) := \texttt{columnOperation}(\texttt{B}, \texttt{R}, \bar{\texttt{R}}, E_{\texttt{k},\texttt{j},-\texttt{q}});$
  endif;
until `divisible`;
return $(\texttt{B}, \texttt{Q}, \bar{\texttt{Q}}, \texttt{R}, \bar{\texttt{R}});$

Proposition 3.47 *Let $A, B \in \mathrm{M}_{m,n}(\mathbf{Z})$ be integer matrices such that B is in Smith form up to the $(k-1)$-st entry for some $k \in \{1, 2, \dots \min(m, n)\}$, $B[k : m, k : n] \neq 0$ and*

$$B = Q^{-1}AR$$

for some $\mathbf{Z}$ invertible matrices $Q \in \mathrm{M}_{m,m}(\mathbf{Z})$ and $R \in \mathrm{M}_{n,n}(\mathbf{Z})$. Algorithm 3.46 applied to $(B, Q, Q^{-1}, R, R^{-1}, k)$ always halts. It returns a matrix B' which is in the Smith form up to the k-th entry. It also returns two pairs $(Q', \bar{Q}')$ and $(R', \bar{R}')$ of mutually inverse $\mathbf{Z}$-invertible matrices such that

$$B' = Q'^{-1}AR' = \bar{Q}'AR'.$$

Proof. Let b_i denote the absolute value of $\texttt{B}[\texttt{k}, \texttt{k}]$ after the call of `moveMinNonzero` on the i-th pass of the **repeat** loop. We will show that the sequence $\{b_i\}$ is strictly decreasing.

Fix $i \in \{1, 2, \dots\}$ such that b_i is not the last value of the sequence. Then on the i-th pass of the **repeat** loop either one of the two **next** statements is executed or the divisibility test fails. In the case the first **next** statement is executed, the subcolumn $\texttt{B}[k+1 : m, k]$ of `B` returned by `partRowReduce` is non zero, therefore by Proposition 3.26 the subcolumn contains elements whoosh absolute value is less than b_i. It follows that $b_{i+1} < b_i$. In the case the other **next** statement is executed, the argument is similar. Thus consider the case when `checkForDivisibility` returns `divisible` equal to `false`. In this case we have $\texttt{B}[\texttt{i}, \texttt{j}] = q\texttt{B}[\texttt{k}, \texttt{k}] + r$ for some integers $q \in \mathbf{Z}$ and $r \in (\,0, \texttt{B}[\texttt{k}, \texttt{k}]\,)$. Moreover, $\texttt{B}[\texttt{k}, \texttt{j}] = \texttt{B}[\texttt{i}, \texttt{k}] = 0$, because `checkForDivisibility` is called only when no **next** statement is executed. After adding the `k`-th row to the `i`-th row, the new value of `B[i, k]` equals $B[\mathtt{k},\mathtt{k}] = B'[\mathtt{k},\mathtt{k}]$. Therefore, after subtracting q-times the `k`-th column from the `j`-th column the value of $B[\mathtt{i},\mathtt{j}]$ becomes r and consequently $b_{i+1} < b_i$. This shows that $\{b_i\}$ is a strictly decreasing sequence of positive integers. Therefore is must be finite and the algorithm must stop.

Next we will show that B' is in the Smith form up to the k-th entry. First observe that by Proposition 3.26 the first $k-1$ rows and columns of B and B' do not differ. Since B' is the value of the `B` variable after the **repeat** loop is left, we must have $B'[k+1:m,k] = 0$, $B'[k,k+1:n] = 0$ and $B'[k,k]$ divides all $B'[i,j]$ for $i,j > k$. In particular B' has the form required in the definition of the Smith form up to the k-th entry. In order to prove (3.14) we only need to show that $B'[k-1,k-1]$ divides $B'[k,k]$, because $B'[i,i] = B[i,i]$ for $i = 1,2,\ldots k-1$. However, since $B[k-1,k-1]$ divides all entries of $B[k:m.k:n]$, by Proposition 3.12 $B[k-1,k-1] = B'[k-1,k-1]$ divides all entries of $B'[k:m.k:n]$. In particular it divides $B'[k,k]$.

The last assertion is the consequence of the respective properties of `partRowReduce`, `partColumnReduce`, `rowOperation` and `columnOperation`. □

The following algorithm is obtained by inductively applying `partSmithForm`.

Algorithm 3.48 Smith algorithm.

```
function smithForm(matrix B)
m := numberOfRows(B);
n := numberOfColumns(B);
Q := Q̄ := identityMatrix(m);
R := R̄ := identityMatrix(n);
s := k := 0;
while B[k + 1 : m, k + 1 : n] ≠ 0 do
    k := k + 1;
    (B, Q, Q̄, R, R̄) := partSmithForm(B, Q, Q̄, R, R̄, k);
    if abs(B[k, k]) = 1 then
        s := s + 1;
    endif;
endwhile ;
return (B, Q, Q̄, R, R̄, s, k);
```

Theorem 3.49 *Algorithm 3.48 always stops. Given a matrix $A \in \mathrm{M}_{m,n}(\mathbf{Z})$ on input it returns a matrix $B \in \mathrm{M}_{m,n}(\mathbf{Z})$, $\mathbf{Z}$-invertible, mutually inverse matrices $Q, \bar{Q} \in \mathrm{M}_{m,m}(\mathbf{Z})$, $R, \bar{R} \in \mathrm{M}_{n,n}(\mathbf{Z})$ and non-negative integers s and k such that*

$$B = Q^{-1}AR = \bar{Q}AR.$$

Furthermore, B has the form

$$B = \left[\begin{array}{ccccc|c} b_1 & & & & & \\ & b_2 & & 0 & & \\ & & \cdot & & & \\ & & & \cdot & & 0 \\ & 0 & & & \cdot & \\ & & & & b_t & \\ \hline & & 0 & & & 0 \end{array}\right], \tag{3.16}$$

where b_i are non-zero integers, $|b_i| = 1$ for $i = 1, 2, \ldots s$ and b_i divides b_{i+1} for $i = 1, 2, \ldots t-1$.

Proof. Observe that the `k` variable is increased on every pass of the **while** loop in Algorithm 3.48. Therefore $\mathtt{B}[\mathtt{k}+1:\mathtt{m}, \mathtt{k}+1:\mathtt{n}])$ must finally become zero, which means that the loop is left and the algorithm halts. Moreover, B is in the form (3.16), because by Propositon 3.47 matrix B is in Smith form up to the k-th entry and $\mathtt{B}[\mathtt{k}+1:\mathtt{m}, \mathtt{k}+1:\mathtt{n}]) = 0$. The rest of the assertion is straightforward. □

Corollary 3.50 *Let A be an $n \times m$ matrix with integer coefficients. By means of elementary row and column operations over $\mathbf{Z}$ it is possible to bring A to the form (3.16), where the b_i are positive integers and b_i divides b_{i+1} for all i. In particular, if $f : G \to G'$ is a homomorphism of finitely generated free abelian groups, then there are bases of G and G' such that the matrix of f with respect to those bases is in the form (3.16).*

The matrix B returned by Algorithm 3.48 is called the *normal form* of A.

It should be emphasized that the problem of reducing a matrix to its Smith normal form is different from the classical problem of diagonalizing a square matrix. In the first case we can have two bases, one for the kernel and another for the image. In the latter, a single basis consisting of eigenvectors is required.

We finish this section by an application of the Smith Normal Form Algorithm to the solvability of linear systems of equations $Ax = b$ under the restriction that all the terms must be integers. Observe that even the question of whether a solution exists is non-trivial.

Consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

This is a diagonal matrix, with non-zero entries in the diagonal. Therefore, $\det A \neq 0$. However, this does not mean that we can necessarily solve the equation $Ax = b$. As an example consider $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. There is no integer valued vector which solves the equation. Before proceeding, the reader might wish to consider what are the set of integer vectors b for which an integer vector solution exists.

However, even though we began on a pessimistic note, Algorithm 3.48 may be used to find integer valued solutions of linear equations, if they exist.

Algorithm 3.51 Linear equation solver.

```
function Solve(matrix A, vector b)
  m := numberOfRows(A);
  (B, Q, Q̄, R, R̄, s, t) := smithForm(A);
  c := Q̄ * b;
  for i := 1 to t do
    if B[i, i] divides c[i] then
      c[i] := c[i]/B[i, i];
    else
      return "Failure";
    endif;
  endfor;
  for i := t + 1 to m do
    if c[i] ≠ 0 then
      return "Failure"
    endif;
  endfor;
  return R * c;
```

Theorem 3.52 *If the equation*

$$Ax = b \tag{3.17}$$

has a solution $x \in \mathbf{Z}^n$ *then Algorithm 3.51 returns one such solution. Otherwise it returns* "Failure".

Proof. By Theorem 3.49 variables $\bar{Q}$, B and R returned by smithForm satisfy $B = \bar{Q}AR$. Since Q and R are $\mathbf{Z}$-invertible with their inverses given by $\bar{Q}$ and $\bar{R}$, equation (3.17) is equivalent to the equation

$$B\bar{R}x = \bar{Q}b. \tag{3.18}$$

Put $c := \bar{Q}b$ and $u := \bar{R}x$. Then equation (3.18) has an integer solution if and only if

$$Bu = c$$

has an integer solution. But since B is in normal Smith form, the latter has an integer solution if and only if $B[i, i]$ divides c_i for $i = 1, 2, \ldots t$ and $u_i = 0$ for $i = t + 1, t + 2, \ldots m$. □

Exercises

3.10 For each matrix A specified below, find its normal form B and two integer matrices P and Q, invertible over $\mathbf{Z}$, such that $QB = AP$. Use the information provided by P and Q for presenting bases with respect to which the normal form is assumed, a basis for $\ker A$, and a basis for $\operatorname{im} A$.

(a) $A = \begin{bmatrix} 6 & 4 \\ 4 & 0 \\ 0 & 6 \end{bmatrix}$

(b) The matrix A in Example 3.41.

(c) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

3.4 Structure of Abelian Groups

By now we have essentially constructed all the tools necessary to compute quotients of finitely generated free abelian groups. Therefore we will construct Algorithm 3.55 which allows one to take finitely generated free abelian groups $H \subset G$ and produce the quotient group G/H. Furthermore, in the next section we will use this algorithm to compute the homology of any finite chain complex. However, before doing so we will prove the Decomposition Theorem for Abelian Groups, Theorem 3.58. As was indicated in the introduction this leads to Corollary 3.1 which provides us with a means of distinguishing the homology groups that will be computed. For the proof of Theorem 3.58 we do not need the full power of Algorithm 3.55. Instead we only need to understand quotients $\mathbf{Z}^m/H$ where H is a subgroup of $\mathbf{Z}^m$. Observe (see Lemma 13.68) that this implies that H is free of rank less than or equal to m. We start from the following example.

Example 3.53 Consider a subgroup H of $\mathbf{Z}^3$ generated by the columns of the matrix

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

investigated earlier in Example 3.42. We want to describe the quotient group $\mathbf{Z}^3/H$ which is actually equal to $\mathbf{Z}^3/\operatorname{im} A$. Note that the columns are not linearly independent, the third one is equal to the first one, so it is redundant for this purpose. We computed the Smith normal form of A,

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

in Example 3.42. This normal form is obtained using the bases consisting of the columns of the matrices

$$R = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

in $\mathbf{Z}^3$ viewed, respectively, as the domain and the target space. We will denote the columns of R and Q, respectively, by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. Since $AR = QB$, we get equations on column vectors

$$\begin{aligned} A\mathbf{r}_1 &= \mathbf{q}_1, \\ A\mathbf{r}_2 &= 2\mathbf{q}_2, \\ A\mathbf{r}_3 &= 0. \end{aligned}$$

Thus $H = \operatorname{im} A = \mathbf{Z}\mathbf{q}_1 \oplus \mathbf{Z}2\mathbf{q}_2$. By using the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for $\mathbf{Z}^3$ we get

$$\frac{\mathbf{Z}^3}{H} = \frac{\mathbf{Z}\mathbf{q}_1 \oplus \mathbf{Z}\mathbf{q}_2 \oplus \mathbf{Z}\mathbf{q}_3}{\mathbf{Z}\mathbf{q}_1 \oplus \mathbf{Z}2\mathbf{q}_2}.$$

Consider the equivalence classes of the generators $\mathbf{q}_i$ of $\mathbf{Z}^3$. Due to the presence of $\mathbf{q}_1$ in the denominator, $[\mathbf{q}_1] = 0$. Due to the presence of $2\mathbf{q}_2$, $[\mathbf{q}_2]$ is a cyclic element of order two, so it generates a cyclic subgroup isomorphic to Z_2. The class $[\mathbf{q}_3]$ generates an infinite cyclic group isomorphic to $\mathbf{Z}$. This suggests that

$$\frac{\mathbf{Z}^3}{H} = \langle [\mathbf{q}_2] \rangle \oplus \mathbf{Z}[\mathbf{q}_3] \cong \mathbf{Z}_2 \oplus Z.$$

A formal argument for this guess is a part of the proof of the next theorem.

In the above example we investigated a quotient group of the form $\mathbf{Z}^n / \operatorname{im} A$. We don't want to limit ourselves to this case because the ultimate goal will be computing homology groups $H_k(X) = \ker \partial_k / \operatorname{im} \partial_{k+1}$. Even if we identify $C_k(X)$ with a power $\mathbf{Z}^p$, we still have a quotient G/H of two different subgroups of $\mathbf{Z}^p$ to consider: $G := \ker \partial_k$ and $H = \operatorname{im} \partial_{k+1} \subset G$. Thus we will assume that we have subgroups $H \subset G \subset \mathbf{Z}^p$ with, possibly, unrelated bases V for H and W for G. We will need to know how to express the elements of V as linear combinations of the elements of W. The algorithm Linear Equation Solver helps us to deal with this more complex situation. As usually, we shall identify the bases V and W with matrices containing their elements as columns.

Before presenting the quotient group algorithm we need the following result which generalizes Proposition 3.5.

Proposition 3.54 *Assume $H \subset \mathbf{Z}^n$ has basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m\}$. Let $\iota_H : H \to \mathbf{Z}^n$ denote the inclusion map. Then*

$$\xi_V^{-1}(x) = Vx. \tag{3.19}$$

and

$$V = A_{\iota_H \xi_V^{-1}}. \tag{3.20}$$

Moreover, if R is a $\mathbf{Z}$-invertible $m \times m$ matrix, then the columns of $V' := VR$ constitute another basis of H and the change of coordinates matrix is

$$A_{\xi_{V'}\xi_V^{-1}} = R^{-1}. \tag{3.21}$$

Proof. To prove (3.19) it is enough to check it on the canonical bases. By the definition of the coordinates isomorphism and (3.3) we have

$$\xi_V^{-1}(\mathbf{e}_i) = \mathbf{v}_i = V\mathbf{e}_i.$$

For the same reason

$$A_{\iota_H \xi_V^{-1}}\mathbf{e}_i = \iota_H(\mathbf{v}_i) = \mathbf{v}_i = V\mathbf{e}_i,$$

which proves (3.20).

To prove (3.21) denote by $\mathbf{v'}_1, \mathbf{v'}_2, \ldots \mathbf{v'}_m$ the columns of V'. Take $\mathbf{v} \in H$. Let $x = (x_1, x_2, \ldots x_m)$ be the coordinates of $\mathbf{v}$ in the basis V and let $\mathbf{y} = (y_1, y_2, \ldots y_m)$ be the coordinates of $\mathbf{v}$ in the basis V'. Then

$$\mathbf{v} = \sum_{i=1}^{m} x_i \mathbf{v}_i = \sum_{i=1}^{m} x_i V\mathbf{e}_i = V(\sum_{i=1}^{m} x_i \mathbf{e}_i) = V\mathbf{x}$$

and

$$\mathbf{v} = \sum_{i=1}^{m} y_i \mathbf{v'}_i = \sum_{i=1}^{m} y_i VR\mathbf{e}_i = VR(\sum_{i=1}^{m} y_i \mathbf{e}_i) = VR\mathbf{y}.$$

Therefore $V(\mathbf{x} - R\mathbf{y}) = 0$. Since the columns of V are linearly independent, $\ker V = 0$ and consequently we get $\mathbf{x} = R\mathbf{y}$ and $\mathbf{y} = R^{-1}\mathbf{x}$. By the definition of the change of coordinates matrix we get (3.21). □

Algorithm 3.55 Quotient group finder.

```
function quotientGroup(matrix V, W)
n := numberOfColumns(V);
matrix A;
for i := 1 to numberOfColumns(V)
    A[i] := Solve(W, V[i]);
endfor;
(B, Q, Q̄, R, R̄, s, t) := smithForm(A);
U := W * Q;
return (U, B, s, t);
```

Theorem 3.56 *Assume H and G are subgroups of $\mathbf{Z}^p$, $H \subset G$ and*

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \subset \mathbf{Z}^p, \ \{\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_m\} \subset \mathbf{Z}^p$$

constitute bases respectively in H and G. Algorithm 3.55 started with $V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n]$ and $W = [\mathbf{w}_1 \ \mathbf{w}_2 \ \ldots \ \mathbf{w}_m]$ returns $m \times p$ matrices U, B and non-negative integers s, t such that if $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_m]$ then

$$[\mathbf{u}_1] = [\mathbf{u}_2] = \ldots = [\mathbf{u}_s] = 0, \tag{3.22}$$

$[\mathbf{u}_i]$ *is of order* B_{ii} *for* $i = s+1, s+2, \ldots t$ *and of infinite order for* $i = t+1, t+2, \ldots m$. *Moreover,*

$$G/H = \bigoplus_{i=s+1}^{t} \langle [\mathbf{u}_i] \rangle \oplus \bigoplus_{i=t+1}^{m} \mathbf{Z}[\mathbf{u}_i]. \tag{3.23}$$

Proof. Let $\iota_H : H \to \mathbf{Z}^p$ and $\iota_G : G \to \mathbf{Z}^p$ denote the inclusion maps. By Proposition 3.54 we have

$$WA_{\xi_W \xi_V^{-1}} = A_{\iota_G \xi_W^{-1}} A_{\xi_W \xi_V^{-1}} = A_{\iota_G \xi_V^{-1}} = A_{\iota_H \xi_V^{-1}} = V.$$

Therefore the matrix equation

$$WX = V$$

has a solution $X = A_{\xi_W \xi_V^{-1}}$, which is the matrix of the inclusion map $\iota : H \to G$ in bases V and W. The solution is unique, because the columns of W are linearly independent, i.e. $\ker W = 0$.

Let A and R be the matrices computed in Algorithm 3.55. Since A solves the equation

$$WA = V,$$

it is the matrix of the inclusion map $\iota : H \to G$ in bases V and W. Put $V' := VR$ and let $\mathbf{v'}_1, \mathbf{v'}_2, \ldots \mathbf{v'}_n$ denote the columns of V'. By Proposition 3.54 and Proposition 3.4 the inclusion map ι has matrix B in bases $\{\mathbf{v'}_1, \mathbf{v'}_2, \ldots \mathbf{v'}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$. In particular $B_{ii}\mathbf{u}_i = \iota(\mathbf{v'}_i) = \mathbf{v'}_i \in H$. Since $|B_{ii}| = 1$ for $i = 1, 2, \ldots s$, we see that $\mathbf{u}_i \in H$ which shows (3.22).

Assume in turn that $k[\mathbf{u}_i] = 0$ for some $i \in \{s+1, s+22, \ldots m\}$. Then $k\mathbf{u}_i \in H$, i.e.

$$k\mathbf{u}_i = \sum_{j=1}^{n} \alpha_j \mathbf{v}'_j = \sum_{j=1}^{n} \alpha_j B_{jj} \mathbf{u}_j.$$

This implies that either $i > t$ and then $k = 0$ or $i \in \{s+1, s+2, \ldots t\}$ and $k = \alpha_i B_{ii}$. As a consequence $[\mathbf{u}_i]$ is of infinite order in the first case and of order B_{ii} in the other case.

It remains to prove (3.23). Let $g \in G$. Then

$$g = \sum_{i=1}^{m} \alpha_i \mathbf{u}_i$$

and by (3.22)

$$[g] = \sum_{i=1}^{m} \alpha_i [\mathbf{u}_i] = \sum_{i=s+1}^{m} \alpha_i [\mathbf{u}_i].$$

Therefore $[g]$ belongs to the right hand side of (3.23). To show the uniqueness of the decomposition assume that also

$$[g] = \sum_{i=s+1}^{m} \beta_i [\mathbf{u}_i].$$

Then

$$\sum_{i=s+1}^{m} (\alpha_i - \beta_i)\mathbf{u}_i \in H,$$

i.e.

$$\sum_{i=s+1}^{m} (\alpha_i - \beta_i)\mathbf{u}_i = \sum_{i=1}^{s} \gamma_i B_{ii} \mathbf{u}_i.$$

This implies $\alpha_i = \beta_i$ for $i = s+1, s+2, \ldots m$. □

The following corollary is obtained from the previous theorem by shifting indices of $[\mathbf{u}_i]$ so to eliminate the first s trivial equivalence classes and by switching the order of finite cyclic with infinite cyclic elements.

Corollary 3.57 *Consider subgroups $H \subset G \subset \mathbf{Z}^p$. Then there exist elements $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_q \in G$ and an integer $r \geq 0$ and $k := q - r$ such that*

(a) The quotient group G/H is generated by equivalence classes $[\mathbf{u}_i]$, $i = 1, 2, \ldots q$;
(b) If $r > 0$, $[\mathbf{u}_i]$ is cyclic of infinite order for $i \in \{1, 2, \ldots r\}$;
(c) If $k = q - r > 0$, $[\mathbf{u}_{r+i}]$ is cyclic of order b_i for $i \in \{1, 2, \ldots k\}$, where $b_i > 1$ and b_i divides b_{i+1} for $i \in \{1, 2, \ldots k-1\}$ provided $k > 1$.
(d) The group G/H can be decomposed as

$$G/H = \bigoplus_{i=1}^{r} \mathbf{Z}[\mathbf{u}_i] \ \oplus \ \bigoplus_{i=r+1}^{q} \langle [\mathbf{u}_i] \rangle.$$

By means of the last theorem we can characterize the general structure of a finitely generated abelian group.

Theorem 3.58 *Let G be a finitely generated abelian group. Then G can be decomposed as a direct sum of cyclic groups. More explicitly, there exist generators $g_1, g_2, \ldots, g_q$ of G and an integer $0 \leq r \leq q$ such that*

(a) $G = \bigoplus_{i=1}^{q} \langle g_i \rangle$,
(b) If $r > 0$, then $g_1, g_2 \ldots, g_r$ are of infinite order,
(c) If $k = q - r > 0$, then $g_{r+1}, g_{r+2}, \ldots, g_{r+k}$ have finite orders $b_1, b_2, \ldots, b_k$, respectively, $b_i > 1$, and b_i divides b_{i+1} for $i \in \{1, 2, \ldots k-1\}$ provided $k > 1$.

The numbers r and $b_1, b_2, \ldots, b_k$ are uniquely determined by G, although generators $g_1, g_2, \ldots, g_q$ are not.

Corollary 3.59 *Given a finitely generated abelian group G there exists a free subgroup F such that*

$$G = F \oplus T(G). \tag{3.24}$$

In particular, $G/T(G)$ is free.

Proof. Using the notation of Theorem 3.58 let

$$F := \bigoplus_{i=1}^{r} \mathbf{Z} g_i \quad T := \bigoplus_{i=1}^{k} \langle g_{r+i} \rangle \ .$$

It is easy to verify that $T = T(G)$, thus $G = F \oplus T(G)$. Therefore, $G/T(G) \cong F$ which is free. □

This presentation of G in (3.24) is called the *normal decomposition* of G. It is worth remarking that F is unique only up to an isomorphism and depends on the choice of generators.

Proof of Theorem 3.58 Let $S := \{s_1, \ldots, s_m\}$ be a set of generators for G. Define $f : \mathbf{Z}^m \to G$ on the canonical basis of $\mathbf{Z}^m$ by $f(\mathbf{e}_i) = s_i$. This is a group homomorphism and so $H := \ker f$ is a subgroup of $\mathbf{Z}^m$. By Theorem 13.43,

$$\bar{f} : \mathbf{Z}^m/H \to G$$

is an isomorphism. Thus to prove the theorem it is sufficient to obtain the desired decomposition for the group $\mathbf{Z}^m/H$. Since $\mathbf{Z}^m$ is a free group, the conclusion follows immediately from Corollary 3.57.

The statement about the uniqueness of the Betti number r and torsion coefficients b_i is not immediate but we leave it as an exercise with the steps of the proof outlined. □

We conclude this section by reminding the reader to check that Corollary 3.1 does indeed follow from Theorem 3.58.

Exercises

3.11 For each matrix A in Exercise 3.10 find the normal decomposition of the group $\mathbf{Z}^m/\operatorname{im} A$ where m is the number of rows of A.

3.12 If $G \simeq \mathbf{Z}_6 \oplus \mathbf{Z}_{20} \oplus \mathbf{Z}_{25}$, find the torsion coefficients and prime power factors of G.

3.13 Find two different free subgroups of $\mathbf{Z} \oplus \mathbf{Z}_2$ complementing the torsion subgroup $\mathbf{Z}_2$.

3.14 **

(a) Let p be a prime and let $d_1, d_2, \ldots, d_m$ be non-negative integers. Show that if
$$G \simeq (\mathbf{Z}_p)^{d_1} \oplus (\mathbf{Z}_{p^2})^{d_2} \oplus \ldots \oplus (\mathbf{Z}_{p^m})^{d_m} \ ,$$
then the integers d_i are uniquely determined by G.
(**Hint:** Consider the kernel of the homomorphism $f_i : G \to G$ given by $f_i(a) := p^i a$. Show that f_1 and f_2 determine d_1. Proceed by induction.)

(b) Let $p_1, p_2, \ldots, p_s$ be a sequence of primes. Generalize (a) to a finite direct sum of terms of the form $(\mathbf{Z}^j_{p_i})^{d_{ij}}$.

(c) Derive the last conclusion of Theorem 5.1 on the uniqueness of the Betti number and torsion coefficients.

3.5 Computing Homology Groups

The previous sections provided us with sufficient background to formulate an algorithm for computing the k-th homology group of a free chain complex. As usual we start from a simple example.

Example 3.60 Consider the abstract chain complex $\mathcal{C}$ defined as follows. The chain groups are $C_0 := \mathbf{Z}^3$, $C_1 := \mathbf{Z}^3$, and $C_k = 0$ for all $k \neq 0, 1$. The only nontrivial boundary map $\partial_1 : C_1 \to C_0$ is given by

$$\partial_1 \mathbf{x} := A\mathbf{x}$$

where A is the matrix from Example 3.53:

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

The group $H_0(\mathcal{C})$ was already computed in Example 3.53. Indeed,

$$H_0(\mathcal{C}) := \ker \partial_0 / \operatorname{im} \partial_1 = \mathbf{Z}^3 / \operatorname{im} A = \langle [\mathbf{q}_2] \rangle \oplus \mathbf{Z}[\mathbf{q}_3] \cong \mathbf{Z}_2 \oplus \mathbf{Z}.$$

where

$$\mathbf{q}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are the columns of the inverse change of coordinates matrix Q in C_0. Next, by Proposition 3.9,

$$H_1(\mathcal{C}) =: \ker \partial_1 / \operatorname{im} \partial_2 = \ker A = R(\ker B)$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the Smith normal form of ∂_1 and

$$R = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the inverse change of coordinates matrix in C_1. Since $\ker B$ is generated by $\mathbf{e}_3$, $R(\ker B)$ is generated by the last column $\mathbf{r}_3$ of R. Thus

$$H_1(\mathcal{C}) = \mathbf{Z}\mathbf{r}_3 \cong \mathbf{Z}.$$

We shall now formulate a general algorithm for finding homology group of any finitely generated free complex $\mathcal{C} = \{C_k, \partial\}$. We begin by assuming that for each C_k we are given a particular basis that we refer to as the canonical basis. This permits us, via the coordinate isomorphism, to identify C_k with $\mathbf{Z}^{p_k}$ for some p_k and to identify the boundary homomorphisms $\partial_k : C_k \to C_{k-1}$ with matrices $D_k := A_{\partial_k}$.

We assume that these matrices are stored in an array of matrices `D`, constituting the input of the following algorithm.

Algorithm 3.61 Homology group of a chain complex.

```
function homologyGroupOfChainComplex( array[ 0 : ] of matrix D)
 array[ −1 : ] of matrix V, W;
for k := 0 to lastIndex(D) do
    (W[k], V[k − 1]) := kernelImage(D[k]);
endfor;
V[lastIndex(D)] := 0;
 array[ 0 : ] of list H;
for k := 0 to lastIndex(D) do
    H[k] := quotientGroup(V[k], W[k]);
endfor;
return H;
```

Theorem 3.62 *Assume $\mathcal{C} = (C_k, \partial_k)$ is a chain complex and algorithm* `homologyGroup` *is called with an array* `D` *such that* `D[k]` *is the matrix of the boundary operator $\partial_k : C_k \to C_{k-1}$ in some fixed bases in C_k. Then it returns an array* `H` *such that* `H[k]` *represents the k-th homology group of $\mathcal{C}$ in the sense of Theorem 3.56.*

Proof. This is a straightforward consequence of the definition of the homology group, Theorem 3.40 and Theorem 3.56. □

Exercises

3.15 For each matrix A in Exercise 3.11 find the homology groups (with identification of generators) of the abstract chain complex $\mathcal{C}$ defined the same way as in Example 3.60, whose only nontrivial boundary map is $\partial_1 = A$.

3.6 Computing Homology of Cubical Sets

Algorithm 3.61 computes the homology groups of an abstract chain complex and we would like to make use of it to compute the homology groups of a cubical set. What is needed is an algorithm which, given a cubical set, computes the associated cubical chain complex, i.e the generators of the groups of chains, and the boundary map.

The first step is to decide on the data structures needed in such an algorithm. For an elementary interval we use

typedef `interval` := **hash**{`endpoint`} **of int**

where `endpoint` is defined by

typedef `endpoint` := (`left`, `right`)

The representation of an elementary cube in R^d is given by an array of d elementary intervals.

typedef `cube` := **array**[1 :] **of** `interval`

A cubical set X is stored as an array of elementary cubes in $\mathcal{K}_{\max}(X)$.

typedef `cubicalSet` := **array**[1 :] **of** `cube`

A chain $c \in C_k(X)$ is represented as a hash whose keys are the elementary cubes in $\mathcal{K}_k(|c|)$ and the value of the hash on an elementary cube $Q \in \mathcal{K}_k(|c|)$ is just $c(Q)$. Of course in the hash we store only those elementary cubes for which the chain has a non zero value.

typedef `chain` := **hash**{`cube`} **of int**

Finally the boundary map $\partial_k : C_k(X) \to C_{k-1}(X)$ is stored as a hash whose keys are the elementary cubes in $\mathcal{K}_k(X)$ and the value of the hash on an elementary cube $Q \in \mathcal{K}_k(X)$ is the chain $\partial_k(\widehat{Q})$ stored again as a hash.

typedef `boundaryMap` := **hash**{`cube`} **of** `chain`

To be able to switch easily between a hash representation and a vector representation of a chain we need two algorithms which implement the coordinate isomorphism and its inverse.

Our first algorithm computes the vector of canonical coordinates of a chain.

Algorithm 3.63 Finding the coordinates vector of a chain

```
function canonicalCoordinates(chain z, cubicalSet K)
vector v;
for i := 1 to lastIndex(K) do
    if defined(z{K[i]}) then
        v[i] := z{K[i]};
    else
        v[i] := 0;
    endif;
endfor;
return v;
```

The following proposition is straightforward.

Proposition 3.64 *Assume Algorithm 3.63 is called with* `K` *containing an array of all elementary cubes in* $\mathcal{K}_k(X)$ *and* **z** *containing a hash representation of a cycle* $z \in Z_k(X)$*. Then it returns an array* **v** *of all coefficients of* z *in the canonical basis* $\widehat{\mathcal{K}}_k(X)$*, i.e. the coordinate vector of* z *in this basis.*

The next algorithm does the inverse: given canonical coordinates of a chain it computes the chain itself.

Algorithm 3.65 Reconstructing a chain from its canonical coordinates

```
function chainFromCanonicalCoordinates(vector v, cubicalSet K)
chain z := ();
for i := 1 to lastIndex(K) do
    if v[i] ≠ 0 then
        z{K[i]} := v[i];
    endif;
endfor;
return z;
```

The following proposition is straightforward.

Proposition 3.66 *Assume Algorithm 3.65 is called with* K *containing an array of all elementary cubes in $\mathcal{K}_k(X)$ and* v *containing the vector of canonical coordinates of a cycle $z \in Z_k(X)$. Then it returns the hash representation of z.*

Our next task is to compute the chain groups $C_k(X)$. Since these are free groups, we can represent them as lists of generators. Therefore we need to compute the families $\mathcal{K}_k(X)$. We will first present an algorithm, which computes the primary faces of an elementary cube. In the case of an elementary interval $I = [k, k+1]$ these are just the two degenerate intervals $[k]$ and $[k+1]$. Let `left` and `right` be the obvious procedures which, given a representation `I` of a nondegenerate interval, return the degenerate intervals built of the left end right endpoint of `I` respectively.

Algorithm 3.67 Primary faces of an elementary cube

```
function primaryFaces(cube Q)
set of cube L := ∅;
for i := 1 to lastIndex(Q) do
    if Q[i]{left} ≠ Q[i]{right} then
        R := Q;
        R[i]{left} := R[i]{right} := Q[i]{left};
        L := join(L, R);
        R[i]{left} := R[i]{right} := Q[i]{right};
        L := join(L, R);
    endif;
endfor;
return L;
```

It is straightforward to check that given on input an elementary cube the algorithm returns the list of all its primary faces. The next algorithm accepts on input a cubical set represented as a list of its maximal faces and uses Algorithm 3.67 to produce the generators of $C_k(X)$. It uses an algorithm `dim`, which given an elementary cube returns the dimension of this cube. (see Exercise 3.16).

Algorithm 3.68 The groups of cubical chains of a cubical set.

```
function cubicalChainGroups(cubicalSet K)
cube Q;
 array[ −1 : ] of list of cube E;
while K ≠ ∅ do
    (Q, K) := cutFirst(K);
    k := dim(Q);
    L := primaryFaces(Q);
    K := union(K, L);
    E[k − 1] := union(E[k − 1], L);
    E[k] := join(E[k], Q);
endwhile ;
return E;
```

Theorem 3.69 *Let X be a cubical set. Assume Algorithm 3.68 is called with* K *containing a list of elementary cubes in $\mathcal{K}_{max}(X)$. Then it returns an array* E *such that for $k = 0, 1, 2, \ldots \dim(X)$ the element* E$[k]$ *is a list which contains all elements of $\mathcal{K}_k(X)$, i.e. the generators of the group $C_k(X)$.*

Proof. We will show first that the algorithm always stops. Let $n := \dim X$ and for a family $\mathcal{X} \subset \mathcal{K}(X)$ let

$$p(\mathcal{X}) := (p_n(\mathcal{X}), p_{n-1}(\mathcal{X}), \ldots, p_0(\mathcal{X})),$$

where

$$p_i(\mathcal{X}) := \text{card } \{Q \in \mathcal{X} \mid \dim Q = i\} .$$

Let $\mathcal{K}_j$ denote the family of cubes represented by the value of the K variable on entering the j-th pass of the **while** loop. Let Q_j denote the elementary cube removed from $\mathcal{K}_j$ on the call to **cutFirst** and put $m_j := \dim Q_j$. The family $\mathcal{K}_{j+1}$ differs from $\mathcal{K}_j$ by one m_j-dimensional cube removed from $\mathcal{K}_j$ and a bunch of $(m_j - 1)$-dimensional cubes added to it. This means that in lexicographical order

$$p(\mathcal{K}_{j+1}) < p(\mathcal{K}_j).$$

Since all vectors $p(\mathcal{K}_j)$ have only non-negative entries, the sequence $\{p(\mathcal{K}_j)\}$ cannot be infinite and the algorithm must stop.

It remains to prove that the k-th element of the returned array E contains exactly the elementary cubes in $\mathcal{K}_k(X)$. Obviously it contains nothing else. To show that it contains all such elementary cubes take $Q \in \mathcal{K}_k(X)$. Then we can choose a sequence of elementary cubes

$$Q_0 \supset Q_1 \supset Q_2 \supset \ldots \supset Q_r = Q$$

such that $Q_0 \in \mathcal{K}_{\max}(X)$ and Q_{i+1} is a primary face of Q_i. We will show by induction that Q_i is on the list E$[\dim Q + r - i]$. Since $Q_0 \in \mathcal{K}_{\max}(X)$, it is added to $\mathtt{E}[\dim Q_0] = \mathtt{E}[\dim Q + r]$ on the pass of the **while** loop on which Q_0 is removed from `K`. Now assume that Q_j is on the list $\mathtt{E}[\dim Q + r - j]$ for some $j < r$. Then Q_{j+1} is added to $\mathtt{E}[\dim Q_{j+1}] = \mathtt{E}[\dim Q + r - (j+1)]$ on the pass of the **while** loop on which Q_j is removed from `K`. This proves that $Q = Q_r$ is on the list $\mathtt{E}[\dim Q]$. □

The next thing we need is an algorithm computing the matrix of the boundary map in the canonical bases. First we present an algorithm which computes the boundary of an elementary chain.

Algorithm 3.70 The boundary operator of an elementary cube.

```
function boundaryOperator(cube Q)
sgn := 1;
chain c := ();
for i := 1 to lastIndex(Q) do
    if Q[i]{left} ≠ Q[i]{right} then
        R := Q;
        R[i]{left} := R[i]{right} := Q[i]{left};
        c{R} := −sgn;
        R[i]{left} := R[i]{right} := Q[i]{right};
        c{R} := sgn;
        sgn := −sgn;
    endif;
endfor;
return c;
```

Theorem 3.71 *Assume Algorithm 3.70 is called with the representation* `Q` *of an elementary cube Q. Then it returns a hash* `c` *representing the chain $\partial\widehat{Q}$.*

Proof. The conclusion follows easily from Proposition 2.37. □

Using Algorithm 3.70 we obtain an algorithm for computing the matrix of the boundary map in a cubical chain complex. Its input is a cubical chain complex represented by the generators of the cubical chain groups and stored in the following data structure.

```
typedef cubicalChainComplex = array[0 : ] of array[ 1 : ] of cube;
```

Algorithm 3.72 The matrix of the boundary operator

```
function boundaryOperatorMatrix(cubicalChainComplex E)
array[ 0 : ] of matrix D;
for k := 0 to lastIndex(E) do
    m := lastIndex(E[k − 1]);
    for j := 1 to lastIndex(E[k]) do
        c := boundaryOperator(E[k][j])
        D[k][1 : m, j] := canonicalCoordinates(c, E[k − 1]);
    endfor;
endfor;
```

```
return D;
```

The proof of the following proposition is straightforward.

Proposition 3.73 *Let X be a cubical set. Assume Algorithm 3.72 is called with an array* `E` *such that* `E`$[k]$ *is a list which contains all elements of $\mathcal{K}_k(X)$, i.e. the generators of the group $C_k(X)$. Then it returns an array* `D` *such that* `D`$[k]$ *is the matrix of $\partial_k^X : C_k(X) \to C_{k-1}(X)$ in the canonical bases.*

We are now almost ready to present the algorithm which, given a cubical set, computes its homology. It is just enough to combine Algorithms 3.68, 3.72 and refalg:homology-group-finder. However the output of Algorithm 3.61 gives only the information about homology in terms of the coordinates and we would prefer to write down exactly the cycles generating the homology groups. For this end we need the following algorithm, which will translate the output of Algorithm 3.61 to concrete chains.

Algorithm 3.74 Generators of homology

```
function generatorsOfHomology( array[ 0 : ] of list H, cubicalChainComplex E)
for k := 1 to lastIndex(H) do
    m := lastIndex(E[k]);
    (U, B, s, t) := H[k];
     array[ 0 : ] hash{generators, orders} HG;
    for j := 1 to lastIndex(E[k]) do
        if j ≤ t then order := "infinity";
        else order := B[j, j] endif;
        c := chainFromCanonicalCoordinates(U[1:m,j],E[k]);
        HG[k]{generators}[j − s] := c;
        HG[k]{orders}[j − s] := order;
    endfor;
endfor;
return HG;
```

Finally we are ready to present the algorithm which was the main goal of this chapter.

Algorithm 3.75 Homology of a cubical set

```
function homology(cubicalSet K)
E := cubicalChainGroups(K);
D := boundaryOperatorMatrix(E);
H := homologyGroupOfChainComplex(D);
H := generatorsOfHomology(H, E);
return H;
```

Theorem 3.76 *Let X be a cubical set. Assume Algorithm 3.75 is called with* `K` *containing a list of elementary cubes in $\mathcal{K}_{max}(X)$. Then it returns an array of hashes* `HG` *such that* `HG[k]{generators}` *is a list $(c_1, c_2, \dots, c_n)$,* `HG[k]{orders}` *is a list $(b_1, b_2, \dots, b_n)$ such that $c_i \in Z_k(X)$ is a cycle of order b_i and*

$$H_k(X) = \bigoplus_{i=1}^{n} \langle [c_i] \rangle. \tag{3.25}$$

Proof. The theorem follows from Proposition 3.73, Theorem 3.69 and Theorem 3.62. □

Exercises

3.7 Preboundary of a Cycle - Algebraic Approach

In Chapter 6 we will discuss the construction of chain selectors of multivalued maps which relies on the solution to the following problem:

> Given an acyclic cubical set X, an integer $k \geq 0$ and a k-dimensional reduced cycle $z \in C_k(X)$ construct a $(k+1)$-dimensional chain $c \in C_{k+1}$ such that
> $$\partial_{k+1} c = z \tag{3.26}$$

Recall from Section 2.7 that the chain groups and boundary maps in all dimensions $k \geq 1$ of the augmented complex

$$\mathcal{C} := \tilde{\mathcal{C}}(X).$$

coincide with those of the cubical complex $\mathcal{C}(X)$, so the only difference between cycles and reduced cycles is in dimension 0. Any 0-dimensional chain in $C_0(X)$ is a cycle, whereas the reduced cycles are precisely those which are in the kernel of the augmentation map ϵ. Recall that the group of reduced cycles is generated by the differences $\widehat{P} - \widehat{Q}$ of elementary 0-chains in $C_0(X)$.

Observe that acyclicity of X means that $H_*(\mathcal{C}) = \tilde{H}_k(X) = 0$ for all $k \geq 0$ and consequently $Z_k(\mathcal{C}) = B_k(\mathcal{C})$. Thus every k-cycle in $\mathcal{C}$ is a boundary, which implies that a solution to (3.26) exists (of course it need not be unique). Any such solution will be called a *preboundary* of z.

In this section we use Theorem 3.52 to present a purely algebraic approach to the solution of this problem. In order to convert the equation (3.26) to a matrix equation let us rewrite it in the form

$$(\xi_k \partial \xi_{k+1}^{-1})(\xi_{k+1} c) = \xi_k z, \tag{3.27}$$

where $\xi_k : C_k(X) \to \mathbf{Z}^{m_k}$ stands for the coordinate isomorphism and m_k is the number of elementary k-dimensional cubes in X. Putting

$$D := A_{\xi_k \partial \xi_{k+1}^{-1}}, \quad \mathbf{x} := \xi_{k+1} c, \quad \text{and} \quad \mathbf{y} := \xi_k z,$$

we obtain the equation

$$D\mathbf{x} = \mathbf{y}.$$

Combining Algorithms 3.63 and 3.65 with Algorithm 3.51 and Algorithm 3.72 we obtain a solution to the preboundary problem in the form of the following algorithm. Note that we assume that the function `dim` used in it accepts as its argument a variable of type `chain` and returns the dimension of the chain stored in this variable (see Exercise 3.17).

Algorithm 3.77 An algebraic solution to the preboundary problem.

```
function preBoundary(chain z, cubicalSet X)
if z = () then return (); endif;
k := dim(z);
E := cubicalChainGroups(X);
D := boundaryOperatorMatrix(E);
y := canonicalCoordinates(z, E[k]);
x := Solve(D, y);
return chainFromCanonicalCoordinates(x, E[k + 1]);
```

Theorem 3.78 *Assume Algorithm 3.77 is called with* `X` *containing a list representation of an acyclic cubical set X and* `z` *containing a hash representation of a reduced cycle $z \in \tilde{Z}_k(X)$ for some $k \geq 1$. Then it returns the hash representation of a preboundary of z, i.e. a chain $c \in C_{k+1}(X)$ such that $\partial c = z$.*

Proof. The theorem follows immediately from Propositions 3.64, 3.66 3.73 and Theorem 3.52 □

Note that the above algorithm could be easily generalized to cycles of any abstract finitely generated chain complex $\mathcal{C}$ which is acyclic, i.e. $H_*(\mathcal{C}) = 0$. Although the algebraic methods are very general and theoretically correct, in practice they may lead to large computations.

In Chapter 7 we will present a more efficient algorithm, which relies on the geometric structure of cubical chain complexes.

Exercises

3.16 Write an algorithm which given an elementary cube represented as an array in data structure `cube` returns the dimension of the cube.

3.17 Write an algorithm which given a chain represented as a hash in data structure `chain` returns the dimension of the chain.
Hint: Let `c` be a variable of type `chain`. Then `c` is a hash. If `c` actually contains a correct chain, then all keys of `c` are elementary cubes of the same dimension and the dimension of these cubes is the dimension of the chain.

4

Chain Maps and Reduction Algorithms

Continuous maps are used to compare topological spaces, linear maps play the same role for vector spaces, and homomorphisms are the tool for comparing abelian groups. It is therefore natural to introduce the notion of a map between chain complexes which will be called a *chain map*. This notion will permit us to compare different chain complexes in the same fashion that homomorphisms allow us to compare abelian groups. A homomorphism of abelian groups is required to preserve group addition. In a chain complex we have an additional operation: taking the boundary of a chain. Therefore, the definition of a chain map will also require that it preserves boundaries.

As we shall see, this gives a new insight into the concept of elementary collapse introduced in Chapter 2 and leads to a more general class of elementary reductions. This in turn leads to additional algorithms for the computation of homology groups. However, the full picture of how topology, algebra and, also, analysis tie together will start to emerge in the next two chapters where we shall discuss continuous maps $f : X \to Y$ between cubical sets X, Y and their effect on homology groups. Chain maps will become a tool in defining a map $f_* : H_*(X) \to H_*(Y)$ induced by f in homology.

4.1 Chain Maps

As was indicated above, we want to introduce homomorphism between chain complexes. Since our goal is to compare the homologies of these complexes, it is important that these homomorphisms induce maps on homology groups. Our discussion at this level is purely algebraic, so we can begin with two abstract chain complexes $\mathcal{C} = \{C_k, \partial_k\}$ and $\mathcal{C}' = \{C'_k, \partial'_k\}$. For each $k \in \mathbf{Z}$, let $\varphi_k : C_k \to C'_k$ be a linear map. Since φ_k is intended to induce a map on homology, $\varphi_{k*} : H_k(\mathcal{C}) \to H_k(\mathcal{C}')$, an obvious question we should ask is: are there any necessary conditions that φ_k must satisfy?

To answer this, we need to begin with the fundamental question of how, given φ_k, can we define φ_{k*}? There does not appear to be much choice. Elements of homology are equivalence classes of cycles. Thus, we should focus our attention on cycles rather than arbitrary chains and consider φ_k applied to cycles. Let us choose $z \in Z_k$ which gives rise to $[z] \in H_k(\mathcal{C})$. If φ_k is to induce a map φ_{k*}, then there must be a relationship between $\varphi_k(z)$ which is an element of C'_k and $\varphi_{k*}([z])$ which is an element of $H_k(\mathcal{C}')$. On this level of generality the only option seems to be the following: $\varphi_{k*}([z])$ must be the homology class generated by $\varphi_k(z)$. Formally, this can be written as

$$\varphi_{k*}([z]) := [\varphi_k(z)]\,. \tag{4.1}$$

But this formula makes sense only if $\varphi_k(z)$ actually generates a homology class, i.e. if $\varphi_k(z)$ is a cycle. More generally, φ_k must map cycles to cycles.

This observation leads to an easily expressed restriction on the φ_k. By definition $z \in Z_k$ implies that $\partial_k z = 0$. In order for (4.1) to make sense we need $\varphi_k(z) \in Z'_k$, or equivalently $\partial'_k\varphi_k(z) = 0$. Since φ_{k-1} is a linear map, $\varphi_{k-1}(\partial_k z) = \varphi(0) = 0$. Combining these statements gives

$$\partial'_k\varphi_k(z) = 0 = \varphi_{k-1}(\partial_k z).$$

$H_k(\mathcal{C})$ and $H_k(\mathcal{C}')$ are abelian groups and therefore we should insist that $\varphi_{k*} : H_k(\mathcal{C}) \to H_k(\mathcal{C}')$ be an abelian group homomorphism, i.e. a linear map. Thus, at the very least $\varphi_{k*}(0) = 0$. But $0 \in H_k(\mathcal{C})$ and $0 \in H_k(\mathcal{C}')$ are really the equivalence classes consisting of boundaries. Thus, φ_k must map boundaries to boundaries.

Again, this observation can be expressed in terms of a simple formula. Let $b \in B_k$, i.e. $b = \partial_{k+1}c$ for some chain $c \in C_{k+1}$. Since boundaries must map to boundaries, $\varphi_k(b) \in B'_k$. But what chain is $\varphi_k(b)$ to be the boundary of? The only one we have at our disposal is c, so the easiest constraint is to ask that $\varphi_k(b) = \partial'_{k+1}(\varphi_{k+1}c)$. But observe that this is equivalent to

$$\partial'_{k+1}\varphi_{k+1}(c) = \varphi_k(b) = \varphi_k(\partial_{k+1}c).$$

As one might have guessed from the time spent discussing it the relationship

$$\partial'\varphi = \varphi\partial$$

is extremely important, and thus, we make the following formal definition.

Definition 4.1 Let $\mathcal{C} = \{C_k, \partial_k\}$ and $\mathcal{C}' = \{C'_k, \partial'_k\}$ be chain complexes. A sequence of homomorphisms $\varphi_k : C_k \to C'_k$ is a *chain map* if for every $k \in \mathbf{Z}$

$$\partial'_k\varphi_k = \varphi_{k-1}\partial_k. \tag{4.2}$$

We will use the notation $\varphi : \mathcal{C} \to \mathcal{C}'$ to represent the collection of homomorphisms $\{\varphi_k : C_k \to C'_k\}$.

A chain map $\varphi : \mathcal{C} \to \mathcal{C}'$ is called a *chain isomorphism*, if for each k the homomorphism φ_k is an isomorphism. The collection of inverse homomorphisms

$\varphi_k^{-1} : C'_k \to C_k$ is denoted by $\varphi^{-1} : \mathcal{C}' \to \mathcal{C}$. We leave it to the reader to check that φ^{-1} is a chain map.

Another way to describe an equality such as (4.2) is through the language of *commutative diagrams* which we will begin to use. More precisely, to say that the diagram

$$\begin{array}{ccc} C_k & \xrightarrow{\varphi_k} & C'_k \\ \Big\downarrow \partial_k & & \Big\downarrow \partial'_k \\ C_{k-1} & \xrightarrow{\varphi_{k-1}} & C'_{k-1} \end{array}$$

commutes is equivalent to saying that $\partial'_k \varphi_k = \varphi_{k-1}\partial_k$.

The following result is essentially a restatement of the introductory comments.

Proposition 4.2 *If* $\varphi : \mathcal{C} \to \mathcal{C}'$ *is a chain map, then*

$$\varphi_k(Z_k) \subset Z'_k := \ker \partial'_k$$

and

$$\varphi_k(B_k) \subset B'_k := \operatorname{im} \partial'_{k+1}$$

for all $k \in \mathbf{Z}$.

Definition 4.3 Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a chain map. Define $\varphi_* : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ by

$$\varphi_*([z]) = [\varphi(z)]$$

where $z \in \mathbf{Z}_k$.

That this map is well defined follows from Proposition 4.2. More precisely, the elements of $H_k(\mathcal{C})$ are the equivalence classes $[z]$ of elements $z \in Z_k$. We need to show that our definition does not depend on the choice of z. By Proposition 4.2, $\varphi_k(z) \in Z'_k$ so $[\varphi_k(z)] \in H_k(\mathcal{C}')$. Now assume that $[z] = [z']$ for some $z' \in Z_k$. Then, $z' = z + b$ where $b \in B_k$. Since φ is a chain map, $\varphi_k b \in B_k(\mathcal{C}')$, and hence,

$$\varphi_*[z'] = [\varphi_k z'] = [\varphi_k(z + b)] = [\varphi_k z + \varphi_k b] = [\varphi_k z] = \varphi_*[z].$$

We shall now give some simple but important examples of chain maps.

Example 4.4 Consider a chain complex $\mathcal{C} = \{C_k, \partial_k\}$ with a subchain complex $\mathcal{C}' = \{C'_k, \partial_k\}$. Let $\iota_k : C'_k \to C_k$ be the inclusion map given by

$$\iota_k c' = c'$$

for every $c' \in C'_k$. Since

$$\partial_k \iota_k c' = \partial_k c' = \iota_{k-1} \partial_k c',$$

ι is a chain map. Therefore,

$$\iota_* : H_*(\mathcal{C}') \to H_*(\mathcal{C})$$

is defined.

On the level of these abstract chain complexes this example may appear trite. However, consider the case where $A \subset X$ are cubical sets in $\mathbf{R}^d$. Then the inclusion map $i : A \hookrightarrow X$ which maps elementary cubes to elementary cubes by $i(Q) = Q$ gives rise to an inclusion map of the chain complexes, $\iota : \mathcal{C}(A) \hookrightarrow \mathcal{C}(X)$, defined by

$$\iota_k \widehat{Q} = \widehat{Q} \quad \text{for all } Q \in \mathcal{K}_k(A).$$

Thus, beginning with a continuous map between two topological spaces, $i : A \hookrightarrow X$, we have arrived at a map between their homology groups $i_* : H_*(A) \to H_*(X)$.

Given this rapid success, the reader might wonder at this point why we wait until Chapter 6 to formalize the process of passing from a continuous map to a map on homology. The reason is simple. The inclusion map maps cubes to cubes, however in the class of continuous maps this is an extremely rare property.

Given this example and the fact that in this chapter we will present several other continuous maps for which we derive the homology map, the reader might wonder why we wait until Chapter 6 to formalize the process of passing from a continuous map to a map on homology. The reason is simple. In all the examples presented here, cubes are mapped to cubes. Continuous functions which possess this property are extremely rare and while it is possible to develop a theory for this restricted class of maps it does not appear to be worth the effort.

The inclusion map maps cubes to cubes, however in the class of continuous maps this is an extremely rare property.

The reader might notice that ι is a monomorphism, and ask whether or not $\iota_* : H_*(A) \to H_*(X)$ must also be a monomorphism. This is false, because a non-trivial cycle in $\mathcal{C}(A)$ might become a boundary in $\mathcal{C}(X)$. For example, $\Gamma^1 \hookrightarrow [0,1]^2$ but $H_1(\Gamma^1) \cong \mathbf{Z}$ and $H_1([0,1]^2) = 0$ therefore ι_* is not a monomorphism in this case.

We leave the proof of the following proposition as an exercise.

Proposition 4.5 *Let $\mathcal{C} = \{C_k, \partial_k\}$ and $\mathcal{C}' = \{C'_k, \partial'_k\}$ be chain complexes. If $\varphi : \mathcal{C} \to \mathcal{C}'$ is a chain map which is an isomorphism, then*

$$\varphi_* : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$$

is an isomorphism.

Example 4.6 Let $X \subset \mathbf{R}^{d-1}$ be a cubical set and $Y := \{0\} \times X \subset \mathbf{R}^d$. Define $\kappa : \mathcal{C}(X) \to \mathcal{C}(Y)$ by

$$\kappa_k(c) := \widehat{[0]} \diamond c.$$

Since any chain in Y must be of the form $\widehat{[0]} \diamond c$, where $c \in \mathcal{C}(X)$, κ is invertible with the inverse $\kappa^{-1}(\widehat{[0]} \diamond c) = c$. Since

$$\partial_k \kappa_k(c) = \widehat{[0]} \diamond \partial_k c = \kappa_{k-1}\partial_k(c),$$

κ is a chain map. Thus, by Proposition 4.5

$$\kappa_* : H_*(X) \to H_*(Y)$$

is an isomorphism.

Observe that, as in the previous example, the chain map κ can be viewed as being generated by a continuous map $j : X \to Y$ given by

$$j(x) = (0, x).$$

We can easily generalize this example as follows. Again, let $X \subset \mathbf{R}^{d-1}$. Define $j : X \to \mathbf{R}^d$, by

$$j(x_1, \ldots, x_{d-1}) = j(x_1, \ldots, x_l, m, x_{l+1}, \ldots, x_{d-1}).$$

If $m \in Z$, then $j(X)$ is a cubical subset of $\mathbf{R}^d$. It is left to the reader to check that it gives rise to a chain isomorphism between $C(X)$ and $C(j(X))$.

Example 4.7 In the previous example we took a cubical set in $\mathcal{R}^{d-1}$ and immersed it in the higher dimensional space $\mathcal{R}^d$. We can also consider translating cubical sets within the same space. To be more precise, let $X \subset \mathbf{R}^d$ be cubical and let $(m_1, \ldots, m_d) \in \mathbf{Z}^d$. Define $t : X \to \mathbf{R}^d$ by

$$t(x_1, \ldots, x_d) = (x_1 + m_1, \ldots, x_d + m_d).$$

Observe that $t(X) \subset \mathbf{R}^d$ is a cubical set.

To obtain a map on homology we need to generate a chain map. Let $Q \in \mathcal{K}_k(X)$, then Q is the product of elementary intervals,

$$Q = \prod_{i=1}^{d} [a_i, b_i]$$

where $b_i - a_i = 1$ or $b_i = a_i$. Define $\tau : \mathcal{C}(X) \to \mathcal{C}(t(X))$ by $\tau_k(\widehat{Q}) := \widehat{P}$ where

$$P := \prod_{i=1}^{d} [a_i + m_i, b_i + m_i].$$

It is left to the reader to check that τ is a chain isomorphism and therefore by Proposition 4.5 to conclude that

$$\tau_* : H_*(X) \to H_*(t(X))$$

is an isomorphism.

In the previous examples, there was a fairly transparent correspondence between the continuous map and the associated chain map. We now turn to a fundamental example where this is not the case.

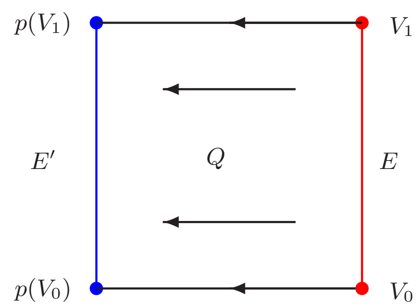


Fig. 4.1. Projection of $Q = [0,1]^2$ and the right edge $E = [1] \times [0,1]$ to the left edge $Q' = E' = [0] \times [0,1]$ described in Example 4.8. Horizontal edges project to vertices so corresponding chains project to 0.

Example 4.8 Consider the elementary cube $Q = [0,1]^d$ and the projection $p : Q \to Q$ given by

$$p(x_1, x_2, x_3, \ldots, x_d) := (0, x_2, x_3, \ldots, x_d).$$

This is illustrated in Figure 4.1 for the case $d = 2$. We want to associate with p a chain map $\pi : \mathcal{C}(Q) \to \mathcal{C}(Q)$ and begin by mimicking the procedure of the previous examples. Any face E of Q can be written as

$$E = E_1 \times E_2 \times E_3 \times \cdots \times E_d$$

where E_i can be either $[0,1]$, $[0]$ or $[1]$. The image of E under p is the elementary cube

$$E' := p(E) = [0] \times E_2 \times E_3 \times \cdots \times E_d$$

If we were to blindly follow what was done before, it would be natural to define $\pi\widehat{E} := \widehat{p(E)}$. However, by definition $\pi_k : C_k(Q) \to C_k(Q)$, but if $E \in \mathcal{K}_k(Q)$ and $E_1 = [0,1]$, then $p(E) \in \mathcal{K}_{k-1}(Q)$. This violates the constraint that k-dimensional chains must be mapped to k-dimensional chains. Therefore, if a nondegenerate interval is projected to a point, the only natural choice for the value of the associated chain map is the trivial chain 0. Thus we define

$$\pi_k(\widehat{E}) := \begin{cases} \widehat{E'} & \text{if } E_1 = [0] \text{ or } E_1 = [1], \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

We will show that π is a chain map. Let $E \in \mathcal{K}_k(Q)$ and decompose it as $E = E_1 \times P$ where $P := E_2 \times E_3 \times \ldots \times E_d$. Using formula (2.6) we obtain

$$\partial\widehat{E} = \partial\widehat{E_1} \diamond \widehat{P} + (-1)^{\dim(E_1)}\widehat{E_1} \diamond \partial\widehat{P}.$$

If $E_1 = [0]$ or $E_1 = [1]$,

$$\begin{aligned} \pi\partial\widehat{E} &= \pi\left(\partial\widehat{E_1} \diamond \widehat{P} + \widehat{E_1} \diamond \partial\widehat{P}\right) \\ &= \pi(\widehat{E_1} \diamond \partial\widehat{P}) \\ &= \widehat{[0]} \diamond \partial\widehat{P} \end{aligned}$$

and consequently

$$\partial\pi\widehat{E} = \partial(\widehat{[0]} \diamond \widehat{P}) = \widehat{[0]} \diamond \partial\widehat{P} = \pi\partial\widehat{E}.$$

If $E_1 = [0,1]$, then $\pi\widehat{E} = 0$ by definition, so $\partial\pi\widehat{E} = 0$. On the other hand

$$\begin{aligned} \pi\partial\widehat{E} &= \pi((\widehat{[1]} - \widehat{[0]}) \diamond \widehat{P} - \widehat{E_1} \diamond \widehat{\partial P}) \\ &= \widehat{[0]} \diamond \widehat{P} - \widehat{[0]} \diamond \widehat{P} - 0 \\ &= 0. \end{aligned}$$

Thus, $\pi : \mathcal{C}(Q) \to \mathcal{C}(Q)$ is a chain map. However, it has additional structure. Observe that on the level of chains it is a projection, i.e. $\pi_k^2 = \pi_k$. To see this, observe that if $E \in \mathcal{K}_k(p(Q))$, then $E = [0] \times E_2 \times \cdots \times E_d$, and hence $\pi_k(\widehat{E}) = \widehat{E}$.

There is another way to interpret the map p. Let $Q' := \{0\} \times [0,1]^{d-1}$. Then $Q' = p(Q)$. For reasons which will become transparent later, we have been careful to distinguish p from the projection $p' : Q \to Q'$ given by the same formula but with Q' as the target space. Similarly, we can define $\pi' : \mathcal{C}(Q) \to \mathcal{C}(Q')$ by the same formula as π, with the only difference being that $\mathcal{C}(Q')$ is the range. The same argument as above shows that π' is a chain projection.

The relation between the two projection maps is $p = ip'$ where $i : Q' \hookrightarrow Q$ is the inclusion map. This brings us back to Example 4.6 and the reader can check that on the chain level

$$\pi_k = \iota_k \circ \pi'_k.$$

In the previous examples, we were able to start with continuous maps and produce maps between the associated homology groups. In some of the examples we were even able to show that we obtained isomorphisms. However, a cautionary note is in order. Consider the simple example of the map $j : [0,1] \to [0,2]$ given by $j(x) = 2x$. Clearly, $H_*([0,1]) \cong H_*([0,2])$ and j is a homeomorphism, however, it is not at all clear how one can pass from j to a chain map $\varphi : C([0,1]) \to C([0,2])$ that, in turn, induces an isomorphism $\varphi_* : H_*([0,1]) \to H_*([0,2])$. This fundamental question will be dealt with in the following three chapters.

Exercises

4.1 Suppose that a chain map $\varphi : \mathcal{C} \to \mathcal{C}'$ is a chain isomorphism, i.e. for each k, the homomorphism φ_k is an isomorphism. Show that φ^{-1} is a chain map.

4.2 Prove Proposition 4.2.

4.3 Show that reflection of Γ^{d-1} about an axis in $\mathbf{R}^d$ gives rise to a chain map on $\mathcal{C}(\Gamma^{d-1})$ which, moreover, is a chain isomorphism.

4.2 Chain Homotopy

We now know that chain maps $\varphi, \psi : \mathcal{C} \to \mathcal{C}'$ generate homology maps $\varphi_*, \psi_* : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$. It is natural to ask what conditions guarantee that $\varphi_* = \psi_*$. To motivate an answer, we return to the setting of cubical complexes. In particular, let X and Y be cubical sets and let $\varphi, \psi : \mathcal{C}(X) \to \mathcal{C}(Y)$.

We shall present a method of linking the two chain maps as restrictions of a chain map on a bigger space. Let us make two copies of X, namely, $\{0\} \times X$ and $\{1\} \times X$. Let $\iota^0 : C(X) \to C(\{0\} \times X)$ and $\iota^1 : C(X) \to C(\{1\} \times X)$ be the inclusion maps described in Example 4.6. Consider the cubical chain map $\chi : C(\{0,1\} \times X) \to C(Y)$ defined by

$$\chi_k(\widehat{[0] \times Q}) = \varphi_k(\widehat{Q}) \tag{4.4}$$

$$\chi_k(\widehat{[1] \times Q}) = \psi_k(\widehat{Q}) \tag{4.5}$$

for all $Q \in \mathcal{K}_k(X)$ and observe that $\varphi = \chi \circ \iota^0$ and $\psi = \chi \circ \iota^1$. Let us assume that χ can be extended to a chain map from $C([0,1] \times X)$ to $\mathcal{C}(Y)$ that satisfies (4.4) and (4.5). For example, in the trivial case that $\psi = \varphi$ the desired extension can be obtained by setting $\chi_{k+1}(\widehat{[0,1] \times Q}) = 0$ for each $Q \in \mathcal{K}_k(X)$. (We leave the verification that this is a chain map as an exercise.)

Let us now make use of the fact that χ is a chain map, i.e. that $\partial\chi = \chi\partial$;

$$
\begin{aligned}
\partial_{k+1}\chi_{k+1}(\widehat{[0,1]} \diamond \widehat{Q}) &= \chi_k \partial_{k+1}(\widehat{[0,1]} \diamond \widehat{Q}) \\
&= \chi_k \left(\widehat{[1]} \diamond \widehat{Q} - \widehat{[0]} \diamond \widehat{Q} - \widehat{[0,1]} \diamond \partial_k \widehat{Q} \right) \\
&= \chi_k \left(\widehat{[1] \times Q} - \widehat{[0] \times Q} - \widehat{[0,1]} \diamond \partial_k \widehat{Q} \right) \\
&= \chi_k \left(\widehat{[1] \times Q} \right) - \chi_k \left(\widehat{[0] \times Q} \right) - \chi_k \left(\widehat{[0,1]} \diamond \partial_k \widehat{Q} \right)
\end{aligned}
$$

Therefore, by (4.4) and (4.5),

$$
\partial_{k+1}\chi_{k+1}(\widehat{[0,1]} \diamond \widehat{Q}) = \psi_k(\widehat{Q}) - \varphi_k(\widehat{Q}) - \chi_k \left(\widehat{[0,1]} \diamond \partial_k \widehat{Q} \right) \tag{4.6}
$$

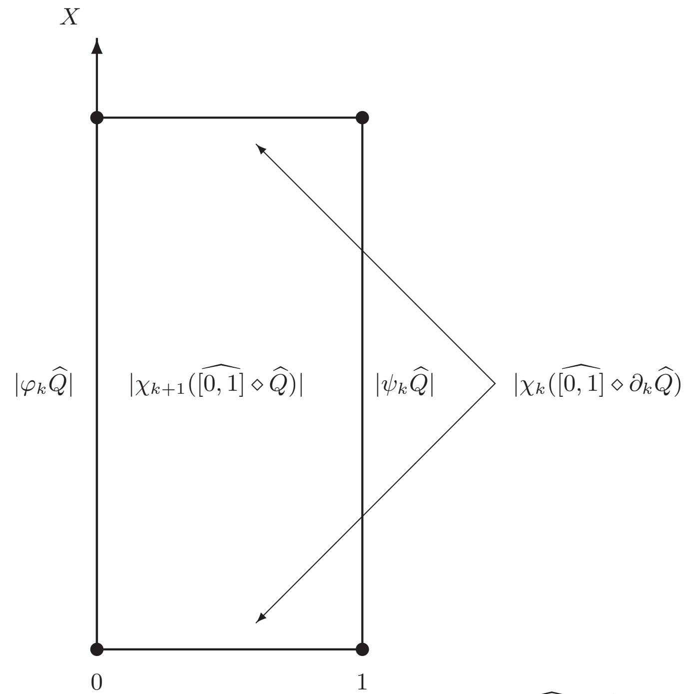


Fig. 4.2. Interpretation of the formula (4.6). The boundary of $\chi_{k+1}(\widehat{[0,1]} \diamond \widehat{Q})$ is composed of three parts: The support of $\varphi_k(\widehat{Q})$ is viewed on the left, the support of $\psi_k(\widehat{Q})$ is viewed on the right, and the support of $\chi_k(\widehat{[0,1]} \diamond \partial_k \widehat{Q})$ is viewed as the lower and upper faces.

A geometric interpretation of the formula (4.6) is indicated in Figure 4.2. If we let

$$
D_k(\widehat{Q}) = \chi_{k+1}(\widehat{[0,1]} \diamond \widehat{Q}).
$$

then (4.6) becomes

$$
\partial_{k+1} D_k(\widehat{Q}) = \psi_k(\widehat{Q}) - \phi_k(\widehat{Q}) - D_{k-1}\left(\partial_k \widehat{Q} \right)
$$

where $D_k : C_k(X) \to C_{k+1}(Y)$ is a homomorphism of groups. This expression is more commonly written as

$$\psi_k - \phi_k = \partial_{k+1} D_k + D_{k-1}\partial_k.$$

Now assume that this relationship is satisfied and apply it to a cycle $z \in C_k(X)$:

$$\begin{aligned}\psi_k(z) - \phi_k(z) &= \partial_{k+1} D_k(z) + D_{k-1}\partial_k(z)\\ &= \partial_{k+1} D_k(z) \in B_k(Y).\end{aligned}$$

Hence, we see that the difference of the values of ψ and ϕ on any cycle is a boundary, and therefore, they induce the same map on homology. This observation leads to the following definition which is presented in the more general setting of abstract chain complexes.

Definition 4.9 Let $\varphi, \psi : \mathcal{C} \to \mathcal{C}'$ be chain maps. A collection of group homomorphisms

$$D_k : C_k \to C'_{k+1}$$

is a *chain homotopy* between φ and ψ if for all $k \in \mathbf{Z}$

$$\partial'_{k+1} D_k + D_{k-1}\partial_k = \psi_k - \varphi_k \ . \tag{4.7}$$

The usefulness of chain homotopy is explained by the following Theorem.

Theorem 4.10 *If there exists a chain homotopy between* φ *and* ψ*, then* $\varphi_* = \psi_*$.

Proof. Let $[z] \in H_k(\mathcal{C})$. Then

$$\begin{aligned}\psi_k(z) - \varphi_k(z) &= \partial'_{k+1} D_k(z) + D_{k-1}\partial_k(z)\\ &= \partial'_{k+1} D_k(z) \in B'_k\end{aligned}$$

Therefore, $[\psi_k(z)] = [\varphi_k(z)]$. □

Example 4.11 We will show that the chain projection $\pi : \mathcal{C}(Q) \to \mathcal{C}(Q)$ in Example 4.8 is chain homotopic to the identity map on $\mathcal{C}(Q)$. Any $E \in \mathcal{K}_k(Q)$ can be written as before as $E = E_1 \times P$, where E_1 is $[0]$, $[1]$, or $[0,1]$ (the last case is excluded if $k = 0$).

Define $D_k : C_k(Q) \to C_{k+1}(Q)$ by

$$D_k(\widehat{E}) := \begin{cases} \widehat{[0,1]} \diamond \widehat{P} & \text{if } E_1 = [1]\\ 0 & \text{if } E_1 = [0]\\ 0 & \text{if } E_1 = [0,1]\end{cases}$$

There is a simple geometric interpretation of this definition. View the corresponding projection $p : Q \to Q$ discussed in Example 4.8 as the final stage $t = 1$ of the deformation $h : Q \times [0,1] \to Q$ given by the formula

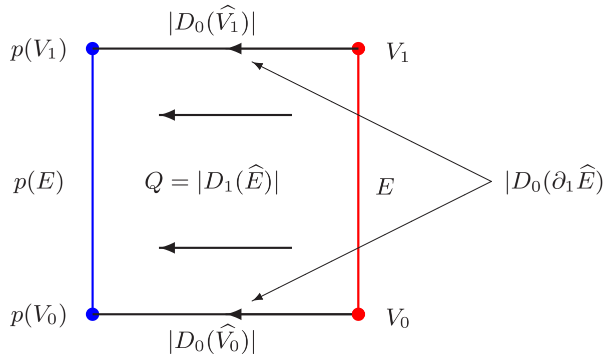


Fig. 4.3. Chain homotopy between the projection π and the identity on $\mathcal{C}(Q)$ in Example 4.11. $\widehat{E} - \pi\widehat{E}$ represents a part of $\partial\widehat{Q}$. $D_{k-1}\partial_k\widehat{E}$ represents the remaining lower and upper faces obtained as images of $\partial\widehat{E}$ under the continuous deformation h.

$$h(x,t) := ((1-t)x_1, x_2, x_3, \ldots, x_d).$$

Recall that E_1 is either $[1]$, $[0]$ or $[0,1]$. In the first case, when $E_1 = [1]$, the support of $\chi_k(\widehat{E})$ is precisely the image $h(E,[0,1])$ of E under the deformation and it represents the face through which E is projected to $E' = [0] \times P$. In the second case, when $E_1 = [0]$, the deformation does not move E, so there is no $(k+1)$-dimensional face involved. In the third case, when $E_1 = [0,1]$, the deformation contracts E_1 to its vertex $[0]$ so, as in the second case, there is no $(k+1)$-dimensional face involved.

We need to verify that condition (4.7) holds for $\psi = \pi$ and $\varphi = \mathrm{id}$. In other words, we must show that for every $E \in \mathcal{K}_k(Q)$

$$\partial_{k+1}D_k\widehat{E} + D_{k-1}\partial_k\widehat{E} = \widehat{E} - \pi\widehat{E}\ . \tag{4.8}$$

Consider the first case in which $E_1 = [1]$. Then

$$\begin{aligned}\partial_{k+1}D_k\widehat{E} + D_{k-1}\partial_k\widehat{E} &= \partial_{k+1}(\widehat{[0,1]} \diamond \widehat{P}) + D_{k-1}(\widehat{[1]} \diamond \partial_k\widehat{P}) \\ &= \widehat{[1]} \diamond \widehat{P} - \widehat{[0]} \diamond \widehat{P} - \widehat{[0,1]} \diamond \partial_k\widehat{P} + \widehat{[0,1]} \diamond \partial_k\widehat{P} \\ &= \widehat{E} - \pi\widehat{E}.\end{aligned}$$

Next let $E_1 = [0]$. Then each term in the left hand side of Equation (4.8) is zero by the definition of D_k and the right hand side is zero because $\widehat{\pi E} = \widehat{E}$. Finally let $E_1 = [0,1]$. Then $D_k\widehat{E} = 0$ so we get

$$\partial_{k+1}D_k\widehat{E} + D_{k-1}\partial_k\widehat{E} = 0 + D_{k-1}(\widehat{[1]} \diamond \widehat{P} - \widehat{[0]} \diamond \widehat{P} - \widehat{[0,1]} \diamond \partial_k\widehat{P})$$

$$
\begin{aligned}
&= \widehat{[0,1]} \diamond \widehat{P} - 0 - 0 \\
&= \widehat{E} \\
&= \widehat{E} - \pi\widehat{E}
\end{aligned}
$$

because $\pi\widehat{E} = 0$.

We have shown that chain maps induce maps on homology and we also have a condition under which different chain maps induce the same map on homology. We now turn to the composition of maps. The proof of the following proposition is left as an exercise.

Proposition 4.12 *Let $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$ be chain complexes. If $\varphi : \mathcal{C} \to \mathcal{C}'$ and $\psi : \mathcal{C}' \to \mathcal{C}''$ are chain maps, then $\psi\varphi : \mathcal{C} \to \mathcal{C}''$ is a chain map and*

$$(\psi\varphi)_* = \psi_*\varphi_*.$$

Eventually we will use homology maps to compare topological spaces. With this in mind it is natural to ask: when does a chain map induce an isomorphism in homology?

Definition 4.13 A chain map $\varphi : \mathcal{C} \to \mathcal{C}'$ is called a *chain equivalence* if there exists a chain map $\psi : \mathcal{C}' \to \mathcal{C}$ such that $\psi\varphi$ is chain homotopic to $\mathrm{id}_{\mathcal{C}}$ and $\varphi\psi$ is chain homotopic to $\mathrm{id}_{\mathcal{C}'}$.

Proposition 4.14 *If $\varphi : \mathcal{C} \to \mathcal{C}'$ is a chain equivalence, then $\varphi_* : H_*(\mathcal{C}) \to H_*(\mathcal{C}')$ is an isomorphism.*

The proof is left as an exercise.

Example 4.15 We continue Examples 4.8 and 4.11 by showing that the projection $\pi' : \mathcal{C}(Q) \to \mathcal{C}(Q')$ and inclusion $\iota : \mathcal{C}(Q') \to \mathcal{C}(Q)$ are chain equivalences. This will imply that $H_*(Q) \cong H_*(Q')$.

Indeed, $\pi'\iota = \mathrm{id}_{\mathcal{C}(Q')}$ so, in particular, $\pi'\iota$ is chain homotopic to $\mathrm{id}_{\mathcal{C}(Q')}$. The reverse composition is $\iota\pi' = \pi$. But Example 4.11 shows that π is chain homotopic to $\mathrm{id}_{\mathcal{C}(Q)}$.

From the above we immediately get another way of proving Theorem 2.76. Since we already have one proof, we shall leave some verifications as exercises.

Corollary 4.16 *The elementary cube $Q = [0,1]^d$ is acyclic.*

Proof. We argue by induction on d. If $d = 0$, Q is a single vertex so it is acyclic by definition. Suppose we know the result for $P = [0,1]^{d-1}$ and consider $Q = [0,1]^d$. By Example 4.15, $H_*(Q) \cong H_*(Q')$, where $Q' = [0] \times P$ is a face of Q. By Example 4.6, $H_*([0] \times P) \cong H_*(P)$, so the conclusion follows. □

Corollary 4.17 *All elementary cubes are acyclic.*

Proof. Let $Q = I_1 \times I_2 \times \cdots \times I_d$ be an elementary cube. Example 4.6 shows that inserting a degenerate interval on a given coordinate does not change homology. Similarly, homology remains the same when degenerate interval is removed. Hence we may assume that all I_i are nondegenerate. Let $I_i = [a_i, a_i + 1]$. Consider the translation map $\tau : \mathcal{C}([0,1]^d) \to \mathcal{C}(Q)$ defined as follows. If $P \in \mathcal{K}_k([0,1]^d)$, then $P = P_1 \times P_2 \times \cdots \times P - d$ where exactly k intervals are nondegenerate. For each k, τ_k is given by

$$\tau_k \widehat{P} := (\widehat{a_1 + P_1}) \diamond (\widehat{a_2 + P_2}) \diamond \cdots \diamond (\widehat{a_d + P_d})$$

where $a_i + P_i$ is the translation of the interval P_i by a_i. This chain map discussed in Example 4.7. As commented there, τ is an isomorphism of chain complexes, so it induces the isomorphism in homology and the conclusion follows from the previous corollary. □

Exercises

4.4 Consider a chain map $\varphi : \mathcal{C}(\Gamma^1) \to \mathcal{C}(\Gamma^1)$ which one can think of as being generated by rotating Γ^1 by 90 degrees in the clockwise direction. More precisely, let $\varphi_0 : C_0(\Gamma^1) \to C_0(\Gamma^1)$ be given on the canonical generators by

$$\begin{aligned}
\varphi_0(\widehat{[0] \times [0]}) &= \widehat{[0] \times [1]} \\
\varphi_0(\widehat{[0] \times [1]}) &= \widehat{[1] \times [1]} \\
\varphi_0(\widehat{[1] \times [1]}) &= \widehat{[1] \times [0]} \\
\varphi_0(\widehat{[1] \times [0]}) &= \widehat{[0] \times [0]}
\end{aligned}$$

and let $\varphi_1 : C_1(X) \to C_1(X)$ be given by

$$\begin{aligned}
\varphi_1(\widehat{[0,1] \times [0]}) &= -\widehat{[0] \times [0,1]} \\
\varphi_1(\widehat{[0] \times [0,1]}) &= \widehat{[0,1] \times [1]} \\
\varphi_1(\widehat{[0,1] \times [1]}) &= -\widehat{[1] \times [0,1]} \\
\varphi_1(\widehat{[1] \times [0,1]}) &= \widehat{[0,1] \times [0]}
\end{aligned}$$

Construct a chain homotopy between φ_* and the identity map on $\mathcal{C}(\Gamma^1)$.

4.5 Prove Proposition 4.12

4.6 Prove Proposition 4.14.

4.7 Consider the cubical cylinder $X = [0,1] \times \Gamma^1 \subset \mathbf{R}^3$ where Γ^1 is the boundary of the unit square. Let $X' = [0] \times \Gamma^1$ be obtained from X by projecting to the plane given by $x_1 = 0$ as in Example 4.8.

(a) Define the corresponding chain projection $\pi : \mathcal{C}(X) \to \mathcal{C}(X)$ with the image $\mathcal{C}(X')$.
(b) Show that π is chain homotopic to identity.
(c) Derive the conclusion that $H_*(X) \cong H_*(\Gamma^1)$.

4.3 Internal Elementary Reductions

In Chapter 3 we presented a purely algebraic algorithm for computing the homology of a chain complex. Unfortunately, if the chain complex is large, then the run time of the algorithm may be very long. On the other hand, in Section 2.4 we introduced the notion of elementary collapse, in part so that we could efficiently compute the homology of some special complexes. Using chain maps we are in a position to generalize the idea of elementary collapse, which has a completely cubical interpretation, to that of an *elementary reduction*, which is based on algebraic considerations. As we shall see this reduction takes the form of a projection of chain groups.

4.3.1 Elementary Collapses Revisited

Our immediate goal is to re-interpret the elementary collapses introduced in Section 2.4 in terms of chain maps.

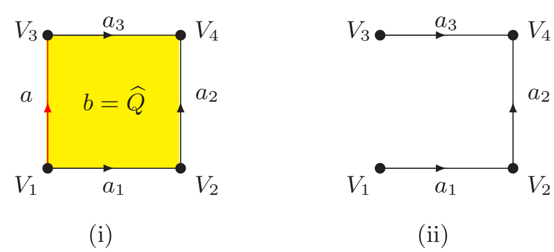


Fig. 4.4. The elementary collapse of a square results in removing a pair of generators (a, b) in dimensions 1 and 2.

Let $\mathcal{C}(Q)$ be the cubical complex of the elementary cube $Q = [0,1]^2$. When we perform the elementary collapse of Q by the edge $[0] \times [0,1]$, two generators are removed from the bases of the chain groups: The generator $a := \widehat{[0] \times [0,1]}$ is removed from the basis $W_1 := \widehat{\mathcal{K}}_1(Q)$ of $C_1(Q)$ and $b = \widehat{Q}$ is removed from the basis $W_2 := \widehat{\mathcal{K}}_2(Q) = \{b\}$. The basis $W_0 := \widehat{\mathcal{K}}_0(Q) = \{\widehat{V}_1, \widehat{V}_2, \widehat{V}_3, \widehat{V}_4\}$ of dual vertices is left untouched. Let us denote by a_1, a_2, and a_3 the remaining generators in W_1. These bases are displayed in Figure 4.4(i).

The chain complex $\mathcal{C}(Q')$ of the collapsed cubical set Q' is indicated in Figure 2.10(ii). Again, explicitly writing out bases for the chain groups we have that $C_0(Q')$ is generated by the basis $W_0' := W_0$, $C_1(Q')$ is generated by the basis $W_1' := W_1 \setminus \{a\} = \{a_1, a_2, a_3\}$, and $C_2(Q')$ is trivial so $W_2' := \emptyset$.

To define a chain map from the complex $\mathcal{C}(Q)$ to $\mathcal{C}(Q')$ we return to topology. Recall Example 2.72 where a deformation retraction of the square onto three of its edges was presented. In this example the edge $|a|$ was mapped onto the three edges $|a_1|$, $|a_2|$ and $|a_3|$. Keeping track of orientations, this suggests that we want to map

$$a \mapsto a' := a_1 + a_2 - a_3.$$

So far we have focused on a single chain, can we describe this operation on the entire chain complex? In other words can we find a chain map $\pi' : \mathcal{C}(Q) \to \mathcal{C}(Q')$ that captures the collapse? The answer is yes and we claim that we have essentially done it. Clearly, $\pi'_2 = 0$. If we order the elements of W_1 as (a_1, a_2, a_3, a) and those of W'_1 as (a_1, a_2, a_3), then

$$\pi'_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \pi'_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We leave it to the reader to check that π' is a chain map, and therefore, induces maps on the associated homology groups.

On the other hand, we are also interested in keeping track of the reduction in terms of the original complex. Observe that $\mathcal{C}(Q')$ is a chain subcomplex of $\mathcal{C}(Q)$ and thus the inclusion map $\iota : \mathcal{C}(Q') \to \mathcal{C}(Q)$ is a chain map.

We would now like to understand the induced maps on homology. In particular, we would like to show that both π' and ι induce isomorphisms on homology. It is easy to check that

$$\pi' \circ \iota = \mathrm{id}_{\mathcal{C}(Q')}$$

and therefore

$$\pi'_* \circ \iota_* = \mathrm{id}_{H_*(Q')}.$$

Let $\pi : \mathcal{C}(X) \to \mathcal{C}(X)$ be defined by $\pi = \iota \circ \pi'$. It is a chain map, since it is the composition of two chain maps. Obviously, $\pi_2 = 0$. For $k = 0, 1$ and using the bases W_k we can write down the following matrices

$$\pi_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.9}$$

π_1 and π_2 are clearly not equal to the identity map. Never the less, we shall show that on the level of homology they induce the identity maps.

By Proposition 4.14 it is sufficient to show that π is chain equivalent to $\mathrm{id}_{\mathcal{C}}$. With this in mind consider the chain homotopy $D_k : C_k \to C_{k+1}$ given by $D_k = 0$ if $k \neq 1$ and

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

Simple matrix calculations show that

$$\pi_1 \circ \mathrm{id}_{C_1} = \partial_2 \circ D_1 + D_0 \circ \partial_1 = \partial_2 \circ D_1.$$

Thus, π induces an isomorphism on homology.

4.3.2 Generalization of Elementary Collapses

As was indicated in the introduction to this section, we would like to be able to reduce the size of the chain complex used to compute homology. In the previous section this was done geometrically using an elementary collapse. Consider, however, an example such as that shown in

Figure 4.5(i). This cubical set is made up of vertices and edges, but has no free faces, and therefore, no elementary collapse is possible.

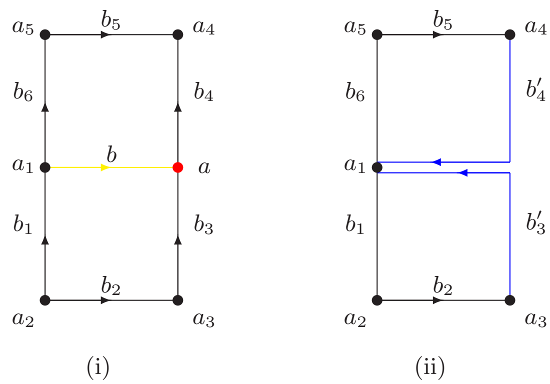


Fig. 4.5. A reduction where an elementary collapse is not possible. The pair of generators (a, b) in dimensions 0 and 1 is removed but also the generators b_3 and b_4 are replaced by new ones.

On the other hand, we saw that when an elementary collapse is written as a chain map, then it can be interpreted as an algebraic operation which takes the form of a projection. With this in mind let us try to mimic the ideas of the previous section to somehow eliminate a pair of k and $k-1$ dimensional chains by directly constructing a projection. To be more precise, let $\mathbf{C} = \{C_k, \partial_k\}$ be an abstract chain complex. Let $b \in C_m$ and $a \in C_{m-1}$. We would like to construct a chain map $\pi : \mathcal{C} \to \mathcal{C}$ which is a projection with the property that $\pi_m(b) = 0$ and $a \notin \operatorname{im} \pi_{m-1}$.

As an analogy to motivate the formula that we will present consider Figure 4.6(a). We want a projection that sends b to 0. V represents the subspace that is perpendicular to b, and hence can be taken to be the image of our projection. What happens to a typical vector q under this projection? As is indicated in Figure 4.6(b), it is sent to the point q' in V. Observe that the relationship between the three vectors is

$$q = q' + \lambda b$$

for some λ. Thus the projection takes the form

$$q \mapsto q' := q - \lambda b,$$

where λ is a scalar depending on q.

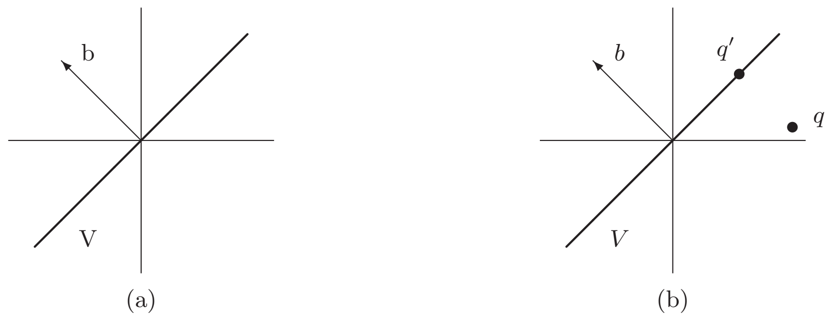


Fig. 4.6. Geometry of a Projection.

Returning to the setting of the desired projection $\pi : \mathcal{C} \to \mathcal{C}$. Let us begin with π_m for $c \in C_m$ set

$$\pi_m(c) = c - \lambda b. \tag{4.10}$$

Of course we still need to determine $\lambda \in \mathbf{Z}$. Recall that we also want to eliminate $a \in C_{m-1}$ from the image of π_{m-1}. In particular, this implies that $0 = \langle \pi_{m-1}\partial c, a\rangle$. Since we want π to be a chain map, we must have

$$\begin{aligned} 0 &= \langle \pi_{m-1}\partial c, a\rangle \\ &= \langle \partial \pi_m c, a\rangle \\ &= \langle \partial(c - \lambda b), a\rangle \\ &= \langle \partial c, a\rangle - \lambda\langle \partial b, a\rangle. \end{aligned}$$

Hence, it must be the case that

$$\lambda = \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}.$$

Observe that since π_k is a group homomorphism, λ must be an integer. Thus we can only define this projection if $\langle \partial b, a\rangle$ divides $\langle \partial c, a\rangle$ for every $c \in C_k$. The easiest way to guarantee this is to choose b and a such that $\langle \partial b, a\rangle = \pm 1$.

To determine π_{m-1}, we begin with the restriction that π is a chain map. In particular, $\pi_m b = 0$ implies that $\partial \pi_m b = 0$, and hence, $\pi_{m-1}\partial b = 0$. Clearly, ∂b needs to be sent to zero under the projection. Thus, following (4.10) we set

$$\pi_{m-1}c = c - \lambda\partial b$$

for any $c \in C_{m-1}$ and again we need to solve for λ. As before, a is not to be in the image of π_{m-1} and hence

$$0 = \langle \pi_{m-1}c, a\rangle = \langle c - \lambda\partial b, a\rangle = \langle c, a\rangle - \lambda\langle \partial b, a\rangle.$$

Solving for λ we obtain

$$\lambda = \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}.$$

Again, the simplest way to insure that λ is an integer is to choose a pair b and a such that $\langle \partial b, a\rangle = \pm 1$.

Putting these calculations together leads to the following definition.

Definition 4.18 Let $\mathcal{C} = \{C_k, \partial_k\}$ be an abstract chain complex. A pair of generators $b \in C_m$ and $a \in C_{m-1}$ with the property that

$$\langle \partial b, a\rangle = \pm 1,$$

is called a *reduction pair* and induces a collection of group homomorphisms

$$\pi_k c := \begin{cases} c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\partial b & \text{if } k = m-1, \\ c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b & \text{if } k = m, \\ c & \text{otherwise,} \end{cases} \tag{4.11}$$

where $c \in C_k$.

Theorem 4.19 *The map $\pi : \mathcal{C} \to \mathcal{C}$ defined by the homomorphisms $\{\pi_k\}_{k\in\mathbf{Z}}$ of (4.11) is a chain map. Furthermore, π_k is a projection of C_k onto its image C'_k, i.e. $\pi_k c = c$ for all $c \in C'_k$.*

Proof. The linearity of π_q is obvious since the maps ∂ and $\langle \cdot, a\rangle$ are linear.

We need to show that $\partial_k\pi_k = \pi_{k-1}\partial_k$ for all k. This is obvious for $k \notin \{m-1, m, m+1\}$.

Let $k = m-1$ and $c \in C_{m-1}$. Then

$$\partial\pi_{m-1}c = \partial\left(c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\partial b\right) = \partial c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\partial^2 b = \partial c$$

because $\partial^2 = 0$. On the other hand $\pi_{m-2}\partial c = \partial c$, because $\pi_{m-2} = \mathrm{id}$.

Let $k = m$ and $c \in C_m$. Then

$$\partial\pi_m c = \partial\left(c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b\right) = \partial c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}\partial b = \pi_{m-1}\partial c\ .$$

Let $k = m+1$ and $c \in C_{m+1}$. Then

$$\pi_m\partial c = \partial c - \frac{\langle \partial^2 c, a\rangle}{\langle \partial b, a\rangle} b = \partial c$$

because $\partial^2 = 0$. On the other hand $\partial \pi_{m+1} c = \partial c$, because $\pi_{m+1} = \mathrm{id}$.

The second conclusion is equivalent to the identity $(\pi_k)^2 = \pi_k$. This is trivial if $k \notin \{m-1, m\}$. Let $k = m-1$ and $c \in C_{m-1}$. Then

$$\begin{aligned}(\pi_{m-1})^2 c &= \pi_{m-1}\left(c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\partial b\right)\\ &= \pi_{m-1}c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\pi_{m-1}\partial b\\ &= \pi_{m-1}c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle}\left(\partial b - \frac{\langle \partial b, a\rangle}{\langle \partial b, a\rangle}\partial b\right)\\ &= \pi_{m-1}c\end{aligned}$$

Let $k = m$ and $c \in C_m$.

$$\begin{aligned}(\pi_m)^2 c &= \pi_m\left(c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b\right)\\ &= \pi_m c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}\pi_m b\\ &= \pi_m c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}\left(b - \frac{\langle \partial b, a\rangle}{\langle \partial b, a\rangle} b\right)\\ &= \pi_m c\end{aligned}$$

□

Example 4.20 Consider the cubical set X presented in Figure 4.5(i). More precisely, let X be the 1-dimensional set consisting of all edges and vertices of the rectangle $R = [0,1] \times [0,2]$. We mentioned earlier that an elementary collapse is not possible in X because it has no free face. Choose a and b, the remaining dual vertices a_1 to a_5, and the remaining dual edges b_1 to b_6 as shown in Figure 4.5(i). The only nontrivial boundary map of $\mathcal{C}(X)$ is ∂_1 defined on the bases $W_0 := \{a_1, a_2, \ldots a_5, a\}$ and $W_1' := \{b_1', b_2', \ldots b_6', b\}$ as follows

$$\begin{aligned}\partial b_1 &= a_1 - a_2,\\ \partial b_2 &= a_3 - a_2,\\ \partial b_3 &= a - a_3,\\ \partial b_4 &= a_4 - a,\\ \partial b_5 &= a_4 - a_5,\\ \partial b_6 &= a_5 - a_1,\\ \partial b &= a - a_1.\end{aligned}$$

Since $\langle \partial b, a\rangle = 1$, (a, b) is a reduction pair. In light of the previous discussion it should be possible to eliminate it from the chain complex. Rather than immediately applying Definition 4.11, we shall go through arguments which lead to it and directly construct the projection. We start from defining π_1 on the generators. To start with, we want to eliminate b, so we define

$$\pi_1 b := 0$$

The definition of π_1 on the remaining generators should permit eliminating a from the boundaries of 1-dimensional chains. This can be done directly by adding or subtracting the above equations. Indeed, a only appears in the equations for the boundary of b_3, b_4 and b. Thus, we get

$$\partial(b_3 - b) = (a - a_3) - (a - a_1) = a_1 - a_3,$$
$$\partial(b_4 + b) = (a_4 - a) + (a - a_1) = a_4 - a_1.$$

This suggests that we should put

$$\pi b_3 = b_3' := b_3 - b,$$
$$\pi b_4 = b_4' := b_4 + b.$$

The remaining generators may be left unchanged:

$$\pi b_i = b_i' := b_i \ , \ i \notin \{3, 4\}.$$

We now define π_0 on generators. We want to have

$$\pi_0 a_i := a_i \ , \ \text{for } i = 1, 2, 3, 4, 5,$$

and to define $\pi_0 a$ so to obtain a chain map. Since $\pi_1 b = 0$, we must have

$$0 = \partial \pi_1 b = \pi_0 \partial b = \pi_0(a) - \pi_0(a_1) = \pi_0(a) - a_1.$$

Therefore, the only choice is

$$\pi_0 a = a' := a_1.$$

It is easily verified that $\partial \pi_1 b_i = \pi_0 \partial b_i$ for all i.

The above formulas coincide with (4.11). As in the case of an elementary collapse discussed in the previous section, we want to look at the reduced complex $\mathcal{C}' := \pi(\mathcal{C}(X))$. Its chain groups C_0' and $\mathcal{C}'$ are generated respectively by

$$W_0' := \{a_1, a_2, \ldots a_5\}$$

and

$$W_1' := \{b_1', b_2', \ldots b_6'\}.$$

The reduced complex $\mathcal{C}'$ is a chain subcomplex of $\mathcal{C}(X))$ but it is not a cubical chain complex because its generators are not elementary dual cubes. Its geometrical interpretation is presented in Figure 4.5(ii). The projection of b to 0 is interpreted as a deformation retraction of the edge $|b|$ to the single point, namely the vertex $|a_1|$. While we deform the edge, the vertex $|a|$ is mapped to $|a_1|$ which makes the definition $\pi(a) := a_1$ clear. While $|a|$ is mapped to $|a_1|$, the adjacent edges b_3 and b_4 can be viewed as being dragged along the edge b thus producing the new generators b_3' and b_4'.

In the above example, the only nontrivial dimensions were 0 and 1, those were the reduction took place. A mistrustful reader may ask what happens if we remove b when there are 2-dimensional cubes in X having b in the boundary. According to Theorem 4.19 the construction should remain correct and this is the case $k = m + 1$ of the proof which shows that. The fact that we replace b_i's by new b_i''s provides a recompense to the missing face b. This is illustrated in the following example.

Example 4.21 Consider the cubical chain complex $\mathcal{C}(R)$ where R is the rectangle $[0,1] \times [0,2]$. The cubical chain complex $\mathcal{C}(X)$ discussed in the previous example was a subcomplex of $\mathcal{C}(R)$ consisting of 0-chains and 1-chains only. Thus $\mathcal{C}(R)$ differs from it only by the group $C_2(R)$ generated by the duals of two squares $Q_1 := [0,1]^2$ and $Q_2 := [0,1] \times [1,2]$ and by the map ∂_2 defined on generators by

$$\partial\widehat{Q}_1 = b_3 - b_1 + b_2 - b,$$
$$\partial\widehat{Q}_2 = b_4 - b_6 + b - b_5.$$

We perform the same reduction of (a, b) as in the previous example. Of course, it is not necessary to do that because R has free edges and we could go with more simple elementary collapses. But we want to see what happens in the highest dimension if we reduce an internal edge and vertex. In terms of the new generators,

$$\partial\widehat{Q}_1 = b_3' - b_1 + b_2,$$
$$\partial\widehat{Q}_2 = b_4' - b_6 - b_5.$$

so b disappears from the boundary expressions. We extend the previous definition of π by putting

$$\pi_2 := \mathrm{id}$$

and we extend the previous definition of the reduced complex by putting $C_2' = C_2(R)$.

We shall now formalize the reduction step discussed in the above examples. Let $(\mathcal{C}, \partial) = (\{C_k\}_{k\in\mathbf{Z}}, \{\partial_k\}_{k\in\mathbf{Z}})$ be a finitely generated free chain complex. Let

$$k_{min}(\mathcal{C}) := \min\{k \mid C_k \neq 0\}$$
$$k_{max}(\mathcal{C}) := \max\{k \mid C_k \neq 0\}$$

For each $k \in [k_{min}(\mathcal{C}), k_{max}(\mathcal{C})]$, let W_k be a fixed basis for C_k. Let $d_k + 1$ be the number of elements in W_k.

Fix $m \in \mathbf{Z}$ and assume that

$$W_{m-1} = \{a_1, a_2, \ldots a_{d_{m-1}}, a\},$$
$$W_m = \{b_1, b_2, \ldots b_{d_m}, b\}$$

where a and b are a reduction pair. Recall that this means that $\langle \partial b, a\rangle = \pm 1$. To simplify the formulae which follow we let

$$r = \partial b - \langle \partial b, a\rangle a.$$

Let $\pi : \mathcal{C} \to \mathcal{C}$ be the associated chain map and let $c' \in C'_k$ denote the image of the chain $c \in C_k$ under π, i.e. $c' := \pi c$. Similarly,

$$C'_k := \pi_k \left(C_k\right).$$

The formula (4.11) is explicit and convenient for some purposes but, since π is a linear map, we would like to express it in terms of generators. Since π_k is the identity in all dimensions other than $m-1$ and m, it is enough to see what are the images of elements of W_{m-1} and W_m under π_{m-1} and π_m respectively. As before we let $c' := \pi c$ for simplicity. It is straightforward to check that

$$a'_i = a_i - \frac{\langle a_i, a\rangle}{\langle \partial b, a\rangle}\partial b = a_i - 0 = a_i,$$

$$a' := a - \frac{\langle a, a\rangle}{\langle \partial b, a\rangle}\partial b = a - \frac{1}{\langle \partial b, a\rangle}(\langle \partial b, a\rangle a + r) = -\frac{r}{\langle \partial b, a\rangle},$$

$$b'_i = b_i - \frac{\langle \partial b_i, a\rangle}{\langle \partial b, a\rangle} b,$$

and

$$b' = b - \frac{\langle \partial b, a\rangle}{\langle \partial b, a\rangle} b = b - b = 0.$$

Thus we get the formulas

$$a'_i = a_i,\ \ a' = -\frac{r}{\langle \partial b, a\rangle} \tag{4.12}$$

and

$$b'_i = b_i - \frac{\langle \partial b_i, a\rangle}{\langle \partial b, a\rangle} b,\ \ b' = 0. \tag{4.13}$$

Define

$$W'_k := \begin{cases} \left\{b'_1, b'_2, \dots, b'_{d_m}\right\} & \text{if } k = m, \\ \left\{a'_1, a'_2, \dots, a'_{d_{m-1}}\right\} & \text{if } k = m-1, \\ W_k & \text{otherwise.} \end{cases} \tag{4.14}$$

We leave as an exercise the proof of the following Proposition.

Proposition 4.22 *W'_k is a basis for C'_k for all $k \in \mathbf{Z}$ and $W'_m \cup \{b\}$ is a basis for C_m.*

Theorem 4.23 *$H_*(\mathcal{C}') \cong H_*(\mathcal{C})$.*

Proof. We will show that $\pi' : \mathcal{C} \to \mathcal{C}'$ is a chain equivalence with the inclusion $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ as a homotopical inverse. Indeed, by Theorem 4.19, π' is a projection so $\pi'\iota = \mathrm{id}_{\mathcal{C}'}$. Hence it is sufficient to find a chain homotopy between $\iota\pi' = \pi$ and $\mathrm{id}_{\mathcal{C}}$. Let $D_k : C_k \to C_{k+1}$ be given by

$$D_k v = \begin{cases} \frac{\langle v,a\rangle}{\langle \partial b,a\rangle} b & \text{if } k = m-1 \ , \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

for any $c \in C_k$. We need to show the identity

$$\mathrm{id}_{C_k} - \pi_k = \partial_{k+1} D_k + D_{k-1}\partial_k \ . \tag{4.16}$$

This is obvious if $k \notin \{m-1, m\}$, because in that case both sides are 0. Let $k = m-1$ and $c \in C_{m-1}$. Then

$$c - \pi_{m-1} c = c - \left(c - \frac{\langle c, a\rangle}{\langle \partial b, a\rangle} \partial b \right) = \frac{\langle c, a\rangle}{\langle \partial b, a\rangle} \partial b \ .$$

On the other hand

$$\partial D_{m-1} c + D_{m-2} \partial c = \partial D_{m-1} c = \partial \left(\frac{\langle c, a\rangle}{\langle \partial b, a\rangle} b \right) = \frac{\langle c, a\rangle}{\langle \partial b, a\rangle} \partial b \ ,$$

so the identity holds.

Let $k = m$ and $c \in C_m$. Then

$$c - \pi_m c = c - \left(c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b \right) = \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b \ .$$

On the other hand

$$\partial D_m c + D_{m-1} \partial c = D_{m-1} \partial c = \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b \ .$$

so again the identity holds. □

Exercises

4.8 Obtain the projection map π and the reduced complex $\mathcal{C}'$ if $\mathcal{C} := \mathcal{C}(\Gamma^2)$ and the reduction pair (a, b) is given by

$$\begin{aligned} a &:= \widehat{[0,1]} \diamond \widehat{[0]} \diamond \widehat{[0]}, \\ b &:= \widehat{[0,1]} \diamond \widehat{[0,1]} \diamond \widehat{[0]}. \end{aligned}$$

4.9 Compute the homology of the complex $\mathcal{C}(X)$ discussed in Example 4.20 by continuing the elementary reductions started there.

4.10 Compute the homology groups of $\mathcal{C} := \mathcal{C}(\Gamma^2)$ by iterated elementary reductions. A reduction pair to start from is given in Exercise 4.8.

4.11 Prove Proposition 4.22.

4.12 Let $\mathcal{C}'$ be the complex obtained from $\mathcal{C}$ by the reduction step. Let $A = (A_{i,j})$ be the matrix of ∂_k with respect to the bases W_k and W_{k-1} and $B = (B_{i,j})$ the matrix of $\partial_{k|\mathcal{C}'}$ with respect to the bases W'_k and W'_{k-1}. Prove the following statements.

(i) If $k \notin \{m-1, m, m+1\}$, then $B = A$.
(ii) If $k = m-1$, then $B = A(1 : d_{m-2}+1, 1 : d_{m-1})$, i.e. B is obtained from A by removing its last column.
(iii) If $k = m$, then B is a $d_{m-1} \times d_m$ matrix (one column and one row less than A) and its coefficients are given by the formula

$$B(i,j) = A(i,j) - \frac{\langle \partial b_j, a\rangle \langle \partial b, a_i\rangle}{\langle \partial b, a\rangle}$$

(iv) If $k = m+1$, then $B = A(1 : d_m, 1 : d_{m+1}+1)$, i.e. B is obtained from A by removing its last row.

4.4 KMS Reduction Algorithm

From the point of view of programming, internal reductions studied in the previous section are more complicated than elementary collapses. This is due to the fact that an elementary collapse can be presented as a removal of a pair of generators from the lists of bases W_{m-1} and W_m while leaving the remaining elements in the list. A general internal reduction require replacing a basis by a different one. The manipulation with data structure could be more simple if we could work with a fixed list of generators as it was the case of collapses. However, if we just remove a pair (a, b) of generators of $\mathcal{C}$, where a is not a free face, the resulting collection of chain groups would not be a chain complex because the boundary operator could be no longer well defined. A solution to this problem is to define a new boundary operator. Those ideas lead to a reduction algorithm due to [29] called *KMS Reduction Algorithm.*

Assume as previously that $(\mathcal{C}, \partial)$ is a finitely generated free abelian chain complex with a fixed basis W_k of C_k and let $(a, b) \in (W_{m-1}, W_m)$ be a reduction pair. In the previous section, we defined a subcomplex $\mathcal{C}'$ of $\mathcal{C}$ having less generators but the same homology as $\mathcal{C}$. The complication comes in the dimension m where the new basis $W'_m = \{b'_1, b'_2, \ldots b'_{d_m}\}$ is related to the basis $W_m = \{b_1, b_2, \ldots b_{d_m}, b\}$ by the formulas

$$b'_i = b_i - \frac{\langle \partial b_i, a\rangle}{\langle \partial b, a\rangle} b. \tag{4.17}$$

We want to define a new chain complex $\bar{\mathcal{C}}$ whose basis would be a subbasis of $\mathcal{C}$. Define

$$\bar{W}_k := \begin{cases} \{b_1, b_2, \ldots, b_{d_m}\} & \text{if } k = m, \\ \{a_1, a_2, \ldots, a_{d_{m-1}}\} & \text{if } k = m-1, \\ W_k & \text{otherwise.} \end{cases} \tag{4.18}$$

Let $\bar{C}_k$ be the free abelian group generated by the set $\bar{W}_k$. Since $\bar{W}_k \subset W_k$, this is a subgroup of C_k but, as we mentioned above, the collection $\bar{\mathcal{C}}$ of those subgroups is not necessarily a subcomplex of $\mathcal{C}$. We want to define a new boundary operator $\bar{\partial}$ so as to give $\bar{\mathcal{C}}$ the structure of a chain complex. The knowledge of the complex $\mathcal{C}'$ from the previous section is useful. We want to assign to the generators b_i of $\bar{C}_m$ the same geometric role which the generators b'_i had in the complex $\mathcal{C}'$. We start with the following example.

Example 4.24 We will review Example 4.20 in the new setting. The complex $\mathcal{C}(X)$ illustrated in Figure 4.5(i) has two nontrivial groups, $C_0(X)$ generated by $W_0 = \{a_1, a_2, \ldots, a_5, a\}$, $C_1(X)$ generated by $W_1 = \{b_1, b_2, \ldots, b_6, b\}$, and one nontrivial boundary map ∂_1 defined on generators by

$$\begin{aligned} \partial b_1 &= a_1 - a_2 \\ \partial b_2 &= a_3 - a_2 \\ \partial b_3 &= a - a_3 \\ \partial b_4 &= a_4 - a \\ \partial b_5 &= a_4 - a_5 \\ \partial b_6 &= a_5 - a_1 \\ \partial b &= a - a_1. \end{aligned}$$

Let $\bar{C}_0$ be the free abelian group generated by $\bar{W}_0 := \{a_1, a_2, \ldots, a_5\}$, and let $\bar{C}_1$ be the free abelian group generated by $\bar{W}_1 := \{b_1, b_2, \ldots, b_6\}$. We already observed that a can be eliminated from the boundaries by subtracting the equation for ∂b from the equation for ∂b_3 and adding it to the equation for ∂b_4. In this way we obtained

$$\begin{aligned} \partial(b_3 - b) &= (a - a_3) - (a - a_1) = a_1 - a_3, \\ \partial(b_4 + b) &= (a_4 - a) + (a - a_1) = a_4 - a_1. \end{aligned}$$

Instead of using those equations to define the new generators b'_3 and b'_4, we use them for defining the new boundary map:

$$\begin{aligned} \bar{\partial}_1 b_3 &:= \partial_1(b_3 - b) = a_1 - a_3, \\ \bar{\partial}_1 b_4 &:= \partial_1(b_4 + b) = a_4 - a_1. \end{aligned}$$

We complete the formula by putting

$$\bar{\pi}_1 b_i := \partial_1 b_i \ , \ i \notin \{3, 4\}.$$

This defines the new complex $\bar{\mathcal{C}}$. It will be shortly seen that the complexes $\bar{\mathcal{C}}$ and $\mathcal{C}'$ are isomorphic.

In order to generalize the above example, consider the collection of homomorphisms $\eta = \{\eta_k : \bar{C}_k \to C'_k\}$ given on any $c \in \bar{C}_k$ by the formula

$$\eta_k(c) := \begin{cases} c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b & \text{if } k = m, \\ c & \text{otherwise.} \end{cases} \tag{4.19}$$

Observe that η_m is given on generators by

$$\eta_m(b_i) = b'_i, \tag{4.20}$$

so it is an isomorphism sending the basis $\bar{W}_m$ to the basis W'_m. When $k \neq m$, η_k is the identity on $\bar{C}_k = C'_k$. The definition of ∂ on $\mathcal{C}'$ may now be transported to $\bar{\mathcal{C}}$ by the conjugacy formula

$$\bar{\partial} := \eta^{-1}\partial\eta \tag{4.21}$$

or, more explicitly,

$$\bar{\partial}_k := \eta_{k-1}^{-1}\partial_k\eta_k. \tag{4.22}$$

Theorem 4.25 *The formula 4.21 defines a boundary map on $\bar{\mathcal{C}}$ thus giving it a structure of chain complex. Moreover, $\eta : \bar{\mathcal{C}} \to \mathcal{C}'$ is a chain isomorphism. In particular, $H_*(\bar{\mathcal{C}}) = H_*(\mathcal{C}') = H_*(\mathcal{C}')$.*

Proof. Since each η_k is an isomorphism, we get

$$\bar{\partial}_{k-1}\bar{\partial}_k = \eta_{k-2}^{-1}\partial_{k-1}\eta_{k-1}\eta_{k-1}^{-1}\partial_k\eta_k = \eta_{k-2}^{-1}\partial_{k-1}\partial_k\eta_k = 0$$

so $\bar{\partial}$ is a boundary map. Next

$$\eta_{k-1}\bar{\partial}_k = \eta_{k-1}\eta_{k-1}^{-1}\partial_k\eta_k = \partial_k\eta_k$$

so η is a chain map. Since each η_k is an isomorphism, the last conclusion is obvious. □

For the purpose of computation, we would like to have an explicit formula for $\bar{\partial}$. This is given by the following

Proposition 4.26 *The following formula holds*

$$\bar{\partial}_k(c) := \begin{cases} \partial c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}\partial b & \text{if } k = m, \\ \partial c - \langle \partial c, a\rangle b & \text{if } k = m+1, \\ \partial c & \text{otherwise.} \end{cases} \tag{4.23}$$

Proof. It is clear from (4.19) and (4.22) that the only dimensions where $\bar{\partial}_k$ may differ from ∂_k are $k = m$ and $k = m + 1$.

Let $k = m$. Then $\eta_{m-1} = \mathrm{id}$ so

$$\bar{\partial}_m c = \partial_m\eta_m c = \partial\left(c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle} b\right) = \partial c - \frac{\langle \partial c, a\rangle}{\langle \partial b, a\rangle}\partial b.$$

Let $k = m + 1$. Then $\eta_{m+1} = \mathrm{id}$ so

$$\bar{\partial}_{m+1} c = \eta_m^{-1} \partial_{m+1} c.$$

Since $c \in \bar{C}_{m+1} = C'_{m+1}$, $\partial c \in C'_m$, so it can be written as

$$\partial_{m+1} c = \sum_i \alpha_i b'_i.$$

But $\eta_m^{-1} b'_i = b_i$ so

$$\bar{\partial}_{m+1} c = \sum_i \alpha_i b_i.$$

On the other hand, letting

$$\alpha = \sum_i \alpha_i \frac{\langle \partial b_i, a \rangle}{\langle \partial b, a \rangle}$$

we obtain

$$\partial_{m+1} c = \sum_i \alpha_i \left(b_i - \frac{\langle \partial b_i, a \rangle}{\langle \partial b, a \rangle} b \right) = \sum_i \alpha_i b_i - \alpha b.$$

From this, we get

$$\langle \partial c, b \rangle = -\alpha$$

so by combining the above formulas we get

$$\bar{\partial} c = \partial c + \alpha b = \partial c - \langle \partial c, b \rangle b.$$

□

In the previous section, the reduced complex $\mathcal{C}'$ was described in terms of the chain projection map $\pi : \mathcal{C} \to \mathcal{C}$ having $\mathcal{C}'$ as the image. It does not make much sense to speak about the chain projection from $\mathcal{C}$ to $\bar{\mathcal{C}}$ because the latter one is not a subcomplex of the first. We may, however, define an analogous chain map $\bar{\pi} : \mathcal{C} \to \bar{\mathcal{C}}$ by the formula

$$\bar{\pi} := \eta^{-1} \pi \tag{4.24}$$

Since π induces an isomorphism in homology and η is a chain isomorphism, the composition $\bar{\pi}$ also induces an isomorphism in homology. Here is an explicit formula for that map.

Proposition 4.27

$$\bar{\pi}_k(c) := \begin{cases} c - \frac{\langle c, a \rangle}{\langle \partial b, a \rangle} \partial b & \textit{if } k = m - 1, \\ c - \langle \partial c, a \rangle b & \textit{if } k = m, \\ c & \textit{otherwise}, \end{cases} \tag{4.25}$$

Proof. If $k \neq m$, η_k is the identity, so the conclusion follows from (4.11).

The proof of the case $k = m$ goes along the same lines as the proof of the case $k = m + 1$ of the Proposition 4.23 and it is left as an exercise. □

Note that the $\bar{\pi}$ gives the mathematical meaning to the KMS elementary reduction step as an operation on chain complexes but it does not explicitly serve to perform the reduction step. This step consists of two operations:

- Remove a reduction pair (a, b) from the lists (W_{m-1}, W_m) of generators;
- Modify the formula for the boundary map on the generators remaining in W_m.

Let now

$$\mathcal{C}^0 \xrightarrow{\bar{\pi}^1} \mathcal{C}^1 \xrightarrow{\bar{\pi}^2} \mathcal{C}^2 \cdots \tag{4.26}$$

be a sequence of chain complexes and projections obtained as follows. Initially set

$$\mathcal{C}^0 := \mathcal{C} \ , \ \partial^0 := \partial \ , \ \text{and } W_k^0 := W_k \ .$$

Then iterate the KMS elementary reduction and define

$$\mathcal{C}^{j+1} := \bar{\mathcal{C}}^j \ , \ W_k^{j+1} := \bar{W}_k^j \ , \ \partial^{j+1} := \bar{\partial}^j$$

as long as it is possible to find a reduction pair in $\bar{\mathcal{C}}^j$. Denote by $M(k) = \sum_k \operatorname{card}(W_k^j)$, for $j = 0, 1, 2, \ldots$. Since $\mathcal{C}$ is finitely generated, $M(j) < \infty$ and $M(j+1) = M(j) - 2$, therefore there exists a final element of that sequence denoted by $(\mathcal{C}^f, \partial^f)$, beyond which the construction cannot be extended.

Corollary 4.28

$$H_*(\mathcal{C}) \cong H_*(\mathcal{C}^f) \ .$$

Proof. The identity follows from Theorem 4.25 by induction. □

Corollary 4.29 *Suppose that the elementary reductions can be successfully performed until $\partial^f = 0$. Then*

$$H(\mathcal{C}) \cong H(\mathcal{C}^f) = \mathcal{C}^f.$$

Proof. The first identity is given by the previous corollary and the second follows from the assumption $\partial^f = 0$ because then

$$H_k(\mathcal{C}) := \ker \partial_k^f / \operatorname{im} \partial_k^f = C_k^f / 0 = C_k^f.$$

□

Observe that if, through a reduction we achieve $\partial^f = 0$, then we have computed $H(\mathcal{C})$. However, even if this is not attained reduction in the number of generators of $\mathcal{C}^f$ speeds up the application of the algorithms presented in Chapter 3.

We are now ready to describe the KMS algorithm based on the described construction. Since we study the case of an abstract chain complex, the generators need not be elementary cubes, they can be anything. Therefore we assume that there is given a data type `generator` and the abstract generators may be any elements of this data type. The chain complex itself will be stored in

```
typedef chainComplex = array[0 : ] of list of generator;
```

For the chains we will use the data structure

```
typedef chain = hash{generator} of int ;
```

and for the boundary map

```
typedef boundaryMap = array[0 : ] of hash{generator} of chain;
```

Algorithm 4.30 Reduction of a pair

```
function reduce(chainComplex E, boundaryMap bd, int i, generator a, b)
for each e in E[i + 1] do
    remove(b, bd[i + 1]{e});
endfor;
for each e in E[i] do
    if a in keys(bd[i]{e}) then
        for each ē in keys(bd[i]{b}) do
            bd[i]{e}{ē} :=
                bd[i]{e}{ē} − bd[i]{b}{a} ∗ bd[i]{e}{a} ∗ bd[i]{b}{ē};
        endfor;
    endif;
endfor;
remove(b, E[i]);
remove(a, E[i − 1]);
remove(b, bd[i]);
remove(a, bd[i − 1]);
return (E, bd);
```

The following proposition is a straightforward consequence of Theorem 4.25.

Proposition 4.31 *Assume Algorithm 4.30 is called with a chain complex represented in* `E` *and* `bd` *and a pair of generators in dimensions* i *and* $i-1$ *constituting a reduction pair. Then it always stops and returns a new chain complex with the same homology as the original one.*

Algorithm 4.32 Reduction of a chain complex

```
function reduceChainComplex(chainComplex E, boundaryMap bd)
for i := lastIndex(E) downto 1 do
    repeat
        found := false;
        LOOP:
        for each b in E[i] do
```

```
            for each a in E[i − 1] do
                if abs(bd[i]{b}a) = 1 then
                    (E, bd) := reduce(E, bd, i, a, b);
                    found := true;
                    break LOOP;
                endif;
            endfor;
        endfor;
    until not found;
endfor;
return (E, bd);
```

Theorem 4.33 *Assume Algorithm 4.32 is called with a chain complex represented in* E *and* bd*. Then it always stops and returns a new chain complex with the same homology as the original one.*

Proof. To see that the algorithm always stops observe that the only loop which might not be completed is the **repeat** loop. For this loop to complete it is necessary that there is no reduction pair found. This must be the case because on every path of this loop the number of generators, where the reduction pair is searched, is smaller, hence in the extreme case there are no generators at all. The rest of the theorem follows from Proposition 4.31 □

Exercises

4.13 Redo Exercise 4.10 in the setting of KMS reductions.

4.14 Derive the analogy of Proposition **??** for the matrix of $\bar{\partial}_k$ with respect to the bases $\bar{W}_k$ and $\bar{W}_{k-1}$.

4.15 Complete the proof of Proposition 4.27.

4.16 Verify that the matrix of $\bar{\partial}_k$ with respect to the bases $\bar{W}_k$ and $\bar{W}_{k-1}$ is identical to the matrix of ∂_k with respect to the bases W'_k and W'_{k-1} discussed in Exercise 4.12.

5

Preview of Maps

Consider two cubical sets X and Y. In Chapter 2 we studied the associated homology groups $H_*(X)$ and $H_*(Y)$. Now assume that we are given a continuous map $f : X \to Y$. It is natural to ask if f induces a group homomorphism $f_* : H_*(X) \to H_*(Y)$? If so, do we get useful information out of it? The answer is yes and we will spend the next two chapters explaining how to define and compute f_*. It is worth noting even at this very preliminary stage that since $H_*(X)$ and $H_*(Y)$ are abelian groups, f_* will in fact be a linear map and therefore from the algebraic point of view easy to use.

In Chapter 4 we already saw that certain simple maps such as the inclusion map $j : A \hookrightarrow X$ of cubical sets do induce chain maps and, consequently, maps in homology. However those maps were very special in the sense that they preserved the cubical structure. The typical continuous map will not have this property. On the other hand, our homology theory is based on cubes. Thus it is essential that we be able to pass from continuous functions to some form of cubical representation. As will be made clear in Chapter 6 on an abstract level this can be done in complete generality. However, in Chapter 7 we will provide algorithms for computing homology maps and for this we will need an explicit procedure for obtaining a cubical representation from a continuous map. There is a variety of ways in which this can be done. Since the focus of this book is on algebraic topology, we chose to use an elementary approximation method known as interval arithmetic. This is presented in Section 5.1 and used in the remaining sections as a way to provide a concrete introduction to the question of how to generate homology maps.

This is a nontrivial problem for which a variety of technical issues need to be overcome. With this in mind we return in Sections 5.2 through 5.4 to the style of Chapter 1. The emphasis is on the big picture. We will introduce ideas and terminology that will be explained and justified in Chapters 6 and 7.

5.1 Rational Functions and Interval Arithmetic

Let $X \subset \mathbf{R}^d$ and $Y \subset \mathbf{R}^{d'}$ be cubical sets and let $f : X \to Y$ be a continuous function. Our eventual goal is to obtain a group homomorphism $f_* : H_*(X) \to H_*(Y)$. The first obvious difficulty is that $H_*(X)$ and $H_*(Y)$ are defined in terms of cubical complexes which are finite combinatorial objects. Thus we need a means of approximating the continuous map f in terms of some finite information.

As was indicated in the introduction we wish to focus is on the issues related to the algebraic topology rather than the approximations. Because of this, for the remainder of this chapter and again in Chapter 7 we restrict our attention to the set of *rational functions*, i.e. maps $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ such that

$$f = (f_1, f_2, \ldots f_{d'})$$

where

$$f_i(x_1, x_2, \ldots x_d) = \frac{p_i(x_1, x_2, \ldots x_d)}{q_i(x_1, x_2, \ldots x_d)}$$

for some polynomials $p_i, q_i : \mathbf{R}^d \to \mathbf{R}$ with integer coefficients. This simple class of maps meets our needs in that it allows us to provide examples for all the essential ideas of homology maps.

The second issue is approximating rational maps in a manner that leads to cubical representations. Here we make use of the concept of *interval arithmetic*. Given two intervals $[k_1, l_1]$ and $[k_2, l_2]$ define the arithmetic operations

$$\begin{aligned}
[k_1, l_1] + [k_2, l_2] &:= [k_1 + k_2, l_1 + l_2] \\
[k_1, l_1] - [k_2, l_2] &:= [k_1 - l_2, l_1 - k_2] \\
[k_1, l_1] * [k_2, l_2] &:= [\min A, \max A] \\
[k_1, l_1]/[k_2, l_2] &:= [\operatorname{floor}(\min B), \operatorname{ceil}(\max B)]
\end{aligned}$$

where

$$\begin{aligned}
A :&= \{k_1 * k_2, k_1 * l_2, l_1 * k_2, l_1 * l_2\} \\
B :&= \{k_1/k_2, k_1/l_2, l_1/k_2, l_1/l_2\}
\end{aligned}$$

and in the last operation it is assumed that $0 \notin [k_2, l_2]$.

A useful feature of interval arithmetic is the following straightforward to prove proposition.

Proposition 5.1 *Assume I and J are two intervals and $\diamond \in \{+, -, *, /\}$ is an arithmetic operation. If $x \in I$ and $y \in J$ then $x \diamond y \in I \diamond J$.*

If $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ is a rational function, it may be extended to intervals by replacing every arithmetic operation on numbers by the corresponding operation on intervals. To be more precise, recall that we are letting $f = (f_1, f_2, \ldots, f_{d'})$ where each $f_i : \mathbf{R}^d \to \mathbf{R}$ is a rational function. Since any rectangle $I_1 \times I_2 \times \cdots \times I_d$ can be identified with a sequence of d intervals $(I_1, I_2, \ldots I_d)$, we can extend f_i to a function $\tilde{f}_i$ which maps elements of $\mathrm{Rect}\,(\mathbf{R}^d)$ to intervals with rational endpoints by replacing every arithmetic operation on numbers by the corresponding operation on intervals. We could now define $\tilde{f} = \tilde{f}_1 \times \tilde{f}_1 \times \cdots \times \tilde{f}_{d'}$. However, the images of $\tilde{f}$ need not lie in $\mathrm{Rect}\,(\mathbf{R}^{d'})$.

For this reason we perform an additional step. Let $Q \in \mathrm{Rect}\,(\mathbf{R}^d)$. Then $\tilde{f}_i(Q) = [q_1, q_2]$. Define $\bar{f}_i : \mathrm{Rect}\,(\mathbf{R}^d) \to \mathrm{Rect}\,(\mathbf{R})$ by $\bar{f}_i(Q) = [\mathrm{floor}\,(q_1), \mathrm{ceil}\,(q_2)]$ and $\bar{f} : \mathrm{Rect}\,(\mathbf{R}^d) \to \mathrm{Rect}\,(\mathbf{R}^{d'})$ by

$$\bar{f} = \bar{f}_1 \times \bar{f}_2 \times \cdots \times \bar{f}_{d'}. \tag{5.1}$$

Theorem 5.2 *If $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ is a rational function and $Q \in \mathrm{Rect}\,(\mathbf{R}^d)$ is a rectangle, then*

$$f(Q) \subset \hat{f}(Q). \tag{5.2}$$

Proof. The theorem follows from Proposition 5.1 by an induction argument on the number of arithmetic operations in the rational function f. □

Formula (5.2) may look strange at the first glance. So it is worth to comment that the left hand side of this formula is the image of a rectangle Q under a rational function f. This image in general is not a rectangle and it may be difficult to find out what exactly the image is. On the other hand the right hand side of the formula is the value of the extension of f to intervals on the intervals whose product is Q. Unlike the left hand side, the right hand side is always a rectangle and it may be easily computed by applying the interval versions of arithmetic operations to arguments. Therefore formula (5.2) may be viewed as an algorithm for finding rectangular upper estimates of the images of rectangles under a rational function.

Exercises

5.2 Maps on an Interval

As was indicated in the introduction the goal of this chapter is to introduce the homology of maps. To keep the technicalities to an absolute minimum, we begin our discussion with maps of the form $f : [a, b] \to [c, d]$. We do this for one very simple reason; we can draw pictures of these functions. This is to help us develop our intuition. In practice we will want to apply these ideas to cubical sets where it is not feasible to visualize the maps, either because the map is too complicated or because the dimension is too high.

With this in mind consider the cubical sets $X = [-2, 2] \subset \mathbf{R}$ and $Y = [-3, 4] \subset \mathbf{R}$, and let $f : X \to Y$ be defined by $f(x) = (x - \frac{4}{3})(x + \frac{4}{5})$. Thus, we have two topological spaces and a continuous map between them. To treat these combinatorially it should come as no surprise that we will use the cubical structure for X and Y, but it remains to be seen how to view f as a combinatorial object.

The first observation we make is that in passing from a continuous function to a combinatorial object we lose information. Consider Figure 5.1 which shows two functions f and g that are close to one another. Since we will only allow ourselves to perform a finite number of computations, we should not necessarily expect to be able to distinguish between f and g. Thus, the information that we will eventually obtain from f_* and g_* is much coarser that the information contained in the maps themselves.

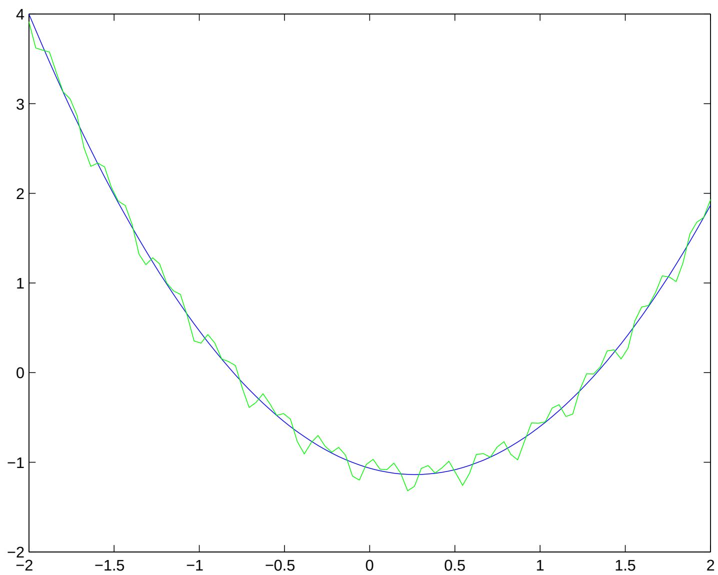


Fig. 5.1. The function $f(x) = (x - \frac{4}{3})(x + \frac{4}{5})$ and a function g which is close in the sense that $|f(x) - g(x)|$ is small for all $x \in [-2, 2]$.

In Chapter 2 we chose elementary cubes as the fundamental objects for determining homology. Clearly,

$$X = [-2, -1] \cup [-1, 0] \cup [0, 1] \cup [1, 2].$$

So let us do our computations in terms of edges. From the combinatorial point of view, this suggests trying to map elementary intervals to sets of intervals. Since $f(-2) = 4$, $f(-1) = \frac{7}{45}$, and f is monotone over the edge $[-2, -1]$, it is clear that

$$f([-2, -1]) \subset [0, 4] = [0, 1] \cup [1, 2] \cup [2, 3] \cup [3, 4].$$

Thus we could think of defining a map that takes the edge $[0,1]$ to the collection of edges $\{[0,1],\ [1,2],\ [2,3],\ [3,4]\}$. Of course, this strategy of looking at the endpoints does not work for the edge $[0,1]$ since f is not monotone here.

It is to deal with this problem that we make use of the interval arithmetic introduced in the previous section. To be more precise, Theorem 5.2 implies that

$$\begin{aligned} f([-1,0]) &\subset \left([-1,0] - \left[\frac{4}{3}, \frac{4}{3}\right]\right) * \left([-1,0] + \left[\frac{4}{5}, \frac{4}{5}\right]\right) \\ &= \left[\frac{-7}{3}, \frac{-4}{3}\right] * \left[\frac{-1}{5}, \frac{4}{5}\right] \\ &= \left[\frac{-28}{15}, \frac{7}{15}\right] \\ &\subset [-2,1] = \bar{f}([-1,0]). \end{aligned}$$

Performing these computations on all the elementary intervals in X leads to Table 5.1.

Edge of X	Bounds on the image	Image Edges
$[-2,-1]$	$\frac{7}{15} \leq f(x) \leq 4$	$\{[0,1],\ [1,2],\ [2,3],\ [3,4]\}$
$[-1,0]$	$-\frac{28}{15} \leq f(x) \leq \frac{7}{15}$	$\{[-2,-1],\ [-1,0],\ [0,1]\}$
$[0,1]$	$-\frac{36}{15} \leq f(x) \leq -\frac{4}{15}$	$\{[-3,-2],\ [-2,-1],\ [-1,0]\}$
$[1,2]$	$-\frac{14}{15} \leq f(x) \leq \frac{28}{15}$	$\{[-1,0],\ [0,1],\ [1,2]\}$

Table 5.1. Elementary edges and their bounding images in terms of sets of edges for $f : X \to Y$.

We can think of Table 5.1 as defining a map from edges to sets of edges. For example

$$[0,1] \mapsto \{[-3,-2], [-2,-1], [-1,0]\}$$

and we can represent this graphically by means of the rectangle

$$[0,1] \times [-3,0] \subset [-2,2] \times [-3,4] = X \times Y.$$

Doing this for all the edges in the domain gives the region shown in Figure 5.2. Observe that the graph of $f : X \to Y$ is a subset of this region and therefore we can think of the region as representing an outer bound on the function f.

We would like to make clearer this idea of mapping edges to sets of edges. Observe that we have started using multivalued maps; that is, functions which take points in X to subsets of Y. Since our goal is to use the computer to perform the necessary computations we do not need to nor want to consider

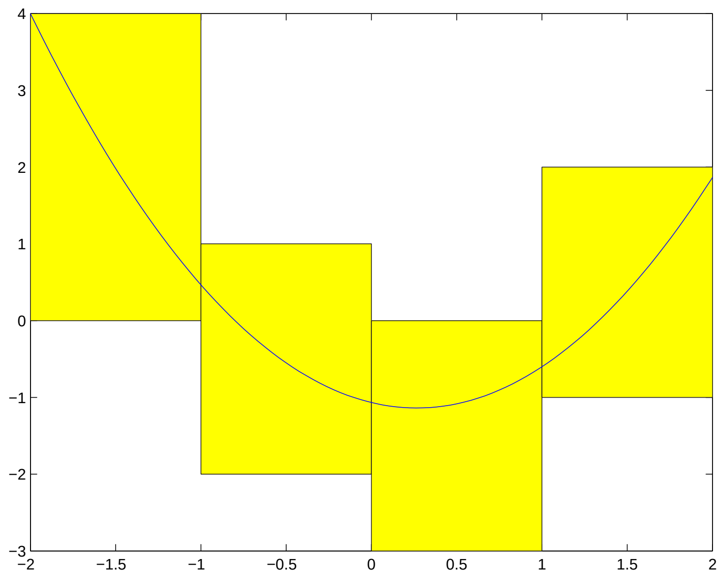


Fig. 5.2. The graph of the map produced by sending edges to sets of edges. Observe that the graph of the function $f(x) = (x - \frac{4}{3})(x + \frac{4}{5})$ lies inside the graph of this edge map.

arbitrary functions of this form. Instead, as we will now indicate for our purposes we will be satisfied with a very special class of multivalued maps.

To derive this class let us think about the essential elements of cubical homology. The first point is that our theory is built on cubical sets. Therefore, it seems reasonable to restrict the images of our multivalued maps to be cubical sets, i.e. given $x \in X$, the image of x should be a cubical set.

Observe however, that using our cubical approach we do not think of the topological space X as being made up of points, but rather elementary cubes. Thus, one might be tempted to add the condition that if x and x' belong to the same elementary cube, then their images should be the same. Unfortunately, this would lead to all maps be constant maps. To see why this is the case consider our example of $X = [-2, 2]$. It contains the elementary cubes $\{[l, l+1] \mid l = -2, -1, \ldots, 1\}$. Since $-1 \in [-2, -1] \cap [-1, 0]$, imposing the above mentioned condition would force the images of all points in $[-2, 0]$ to be the same. By induction we would get that the image of every point in $[-2, 2]$ would have to be the same.

We can, however, avoid this problem by using elementary cells rather than cubes. Recall that the elementary cells which make up $[-2, 2]$ are $\{(l, l+1) \mid l = -2, - \ldots, 1\}$ and $\{[l] \mid l = -2, \ldots, 2\}$ and so different cells do not intersect.

Combining these ideas leads to the following definition.

Definition 5.3 Let X and Y be cubical sets. A multivalued map $F : X \rightrightarrows Y$ is *cubical* if:

1. For every $x \in X$, $F(x)$ is a cubical set.
2. For every $Q \in \mathcal{K}(X)$, $F \mid_{\overset{\circ}{Q}}$ is constant, i.e. if $x, x' \in \overset{\circ}{Q}$, then $F(x) = F(x')$.

Using elementary cells and the edge mapping of Table 5.1 we define the multivalued map

$$F : [-2, 2] \rightrightarrows [-2, 4]$$

by

$$F(x) := \begin{cases} [0,4] & \text{if } x = -2 \\ [0,4] & \text{if } x \in (-2,-1) \\ [0,1] & \text{if } x = -1 \\ [-2,1] & \text{if } x \in (-1,0) \\ [-2,0] & \text{if } x = 0 \\ [-3,0] & \text{if } x \in (0,1) \\ [-1,0] & \text{if } x = 1 \\ [-1,2] & \text{if } x \in (1,2) \\ [-1,2] & \text{if } x = 2 \end{cases}$$

There are four observations to be made at this point. First, for every $x \in X$,

$$f(x) \in F(x).$$

Thus, the cubical multivalued map F acts as an outer approximation for the continuous function f.

Second, F is defined in terms of the vertices and the interior of the edges, i.e. the edges without its endpoints. These are precisely the elementary cells which make up X.

The third observation, is that we used the edges to define the images of the vertices. In particular, we used the formula that if v is a vertex that is a face of edges E_1 and E_2, then

$$F(v) := F(\overset{\circ}{E}_1) \cap F(\overset{\circ}{E}_2). \tag{5.3}$$

The final point is that even though $F : X \rightrightarrows Y$ is a map that is defined on uncountably many points, it is completely characterized by its values on the four edges that make up X. Thus, F is a finitely representable map. This is important because it suggests that it can be stored and manipulated by the computer. We will refer to F as a *representation* of f.

Since the computer cannot work with subsets of $\mathbf{R}$, but rather can only manipulate finite sets, the following concept will play an essential role.

Definition 5.4 Let X and Y be cubical sets. A *combinatorial cubical multivalued map* $\mathcal{F} : \mathcal{K}_{\max}(X) \rightrightarrows \mathcal{K}(Y)$ (see Definition 2.7) is a function from $\mathcal{K}_{\max}(X)$ to subsets of $\mathcal{K}(Y)$.

In particular our map F may be easily encoded in the computer as the combinatorial multivalued map $\mathcal{F} : \mathcal{K}_1([-2,2]) \rightrightarrows \mathcal{K}_1([-3,4])$ given by

$$\begin{aligned}
\mathcal{F}([-2,-1]) &:= \{[0,1],[1,2],[2,3],[3,4]\} \\
\mathcal{F}([-1,0]) &:= \{[-2,-1],[-1,0],[0,1]\} \\
\mathcal{F}([0,1]) &:= \{[-3.-2],[-2,-1],[-1,0]\} \\
\mathcal{F}([1,2]) &:= \{[1-,0],[0,1],[1,2]\}
\end{aligned}$$

Edge of X	Bounds on the image	Image Edges
$[-2,-1.5]$	$1.5 \leq f(x) \leq 4$	$\{[1.5,2],\ [2,2.5],\ [2.5,3],\ [3,3.5],\ [3.5,4]\}$
$[-1.5,-1]$	$0 \leq f(x) \leq 2$	$\{[-0.5,0],\ [0,0.5],\ [0.5,1],\ [1,1.5],\ [1.5,2]\}$
$[-1,-0.5]$	$-1 \leq f(x) \leq 0.5$	$\{[-1,-0.5],\ [-0.5,0],\ [0,0.5]\}$
$[-0.5,0]$	$-1.5 \leq f(x) \leq 0$	$\{[-1.5,-1],\ [-1,-0.5],\ [-0.5,0]\}$
$[0,0.5]$	$-2 \leq f(x) \leq -0.5$	$\{[-2,-1.5],\ [-1.5,0],\ [0,0.5]\}$
$[0.5,1]$	$-1.5 \leq f(x) \leq 0$	$\{[-1.5,-1],\ [-1,-0.5],\ [-0.5,0]\}$
$[1,1.5]$	$-1 \leq f(x) \leq 0.5$	$\{[-1,-0.5],\ [-0.5,0],\ [0,0.5]\}$
$[1.5,2]$	$0 \leq f(x) \leq 2$	$\{[0,0.5],\ [0.5,1],\ [1,1.5],\ [1.5,2]\}$

Table 5.2. Edges and Vertices for the graphs of $X = [-2,2]$ and $Y = [-3,4]$.

The multivalued map F that we constructed above is fairly coarse. If we want a tighter approximation, then one approach is to use finer graphs to describe X and Y. For example, let us write

$$X = \bigcup_{i=0}^{8} [-2 + \frac{i}{2}, -1.5 + \frac{i}{2}] \quad \text{and} \quad Y = \bigcup_{i=0}^{14} [-3 + \frac{i}{2}, -2.5 + \frac{i}{2}]$$

Again, using interval arithmetic we obtain the data described in Table 5.2. The graph of the corresponding multivalued map is shown in Figure 5.3. Observe that this is a tighter approximation to the function than what was obtained with intervals of unit length. In fact, one can obtain as good an approximation as one likes by choosing the edge lengths sufficiently small. In Figure 5.4 one sees the graph of the multivalued map when the lengths of the edges is 0.05.

Of course, this method of subdividing the domain produces tighter and tighter approximations, but unfortunately it takes us out of the class of cubical sets. To avoid this we will adopt an alternative approach. Observe that subdividing has the effect of producing intervals that as individual sets make up a smaller fraction of the domain. We can obtain the same effect by scaling the size of the domain. To be more precise notice that a unit interval in $X = [-2,2]$ represents a quarter of the domain. Now consider the map

$$\Lambda^{(2)} : \mathbf{R} \to \mathbf{R}$$

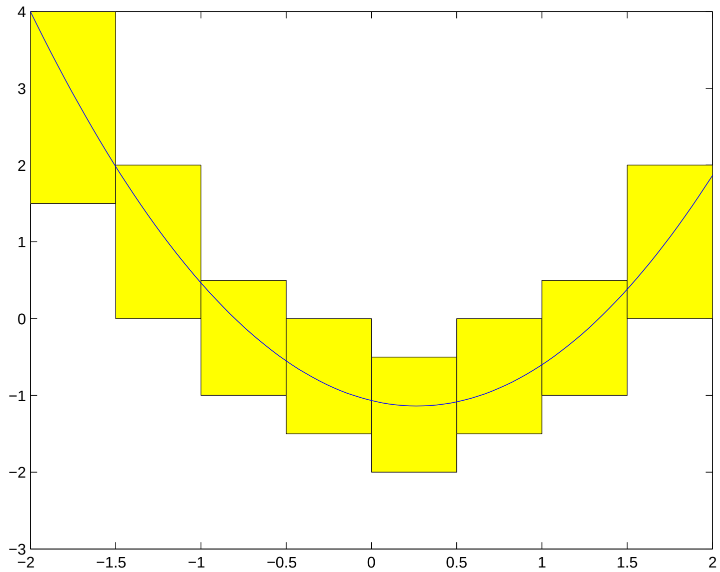


Fig. 5.3. The graph of the multivalued approximation to $f(x) = (x - \frac{4}{3})(x + \frac{4}{5})$ with edges of length 0.5.

given by $\Lambda^{(2)}(x) = 2x$. Define

$$X^{(2)} = \Lambda^{(2)}(X) = [-4, 4]$$

and observe that a unit interval now makes up only an eighth of $X^{(2)}$. Of course, topologically $X^{(2)}$ and X are equivalent since $\Lambda^{(2)}$ has an inverse $\Omega_X^{(2)} : X^{(2)} \to X$ given by

$$\Omega_X^{(2)}(x) = \frac{1}{2}x.$$

It should not be forgotten that our interest is in obtaining a tighter approximation to the map $f : X \to Y$. With this in mind consider $f^{(2)} : X^{(2)} \to Y$ defined by

$$f^{(2)} := f \circ \Omega_X^{(2)}$$

or equivalently $f^{(2)} : [-4, 4] \to Y$ given by

$$f^{(2)}(x) = \left(\frac{x}{2} - \frac{4}{3}\right)\left(\frac{x}{2} + \frac{4}{5}\right).$$

The graph of the associated cubical multivalued map $F^{(2)}$ is indicated in Figure 5.5.

There is no reason that one has to limit the scaling to a factor of 2. More generally, for any integer α we can define $\Lambda^{(\alpha)} : \mathbf{R} \to \mathbf{R}$ by

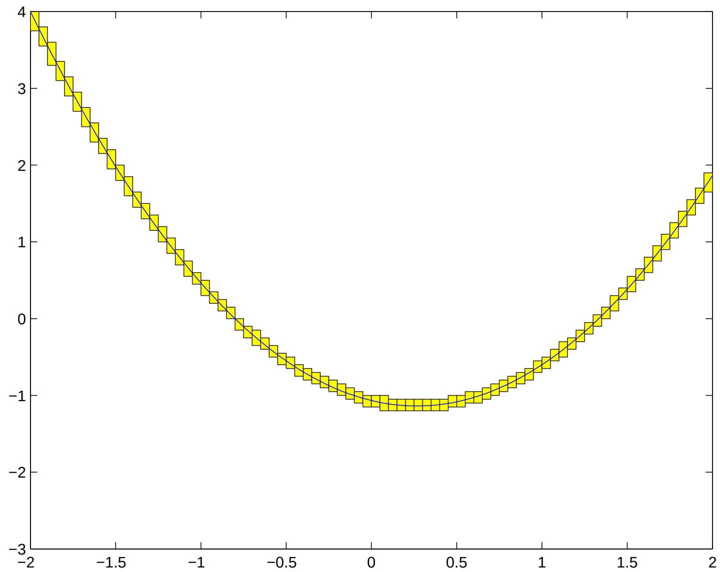


Fig. 5.4. The graph of the multivalued approximation to $f(x) = (x - \frac{4}{3})(x + \frac{4}{5})$ with edges of length 0.05.

$$\Lambda^{(\alpha)}(x) = \alpha x.$$

The reason for insisting that α should be an integer is that this way we are guaranteed that elementary intervals are sent to cubical sets. Under this scaling $X^{(\alpha)} := \Lambda^{(\alpha)}(X) = [-2\alpha, 2\alpha]$. The rescaling of the domain X of f leads to a rescaling of f by means of the inverse map $\Omega_X^{(\alpha)} : X^{(\alpha)} \to X$ defined by $\Omega_X^{(\alpha)}(x) = x/\alpha$. We set

$$f^{(\alpha)} := f \circ \Omega_X^{(\alpha)} : X^{(\alpha)} \to Y.$$

More explicitly, given $x \in X^{(\alpha)}$,

$$f^{(\alpha)}(x) = f\left(\frac{x}{\alpha}\right).$$

Note that we can recover the original map f from $f^{(\alpha)}$ by the formula

$$f(x) = f^{(\alpha)} \circ \Lambda^{(\alpha)}(x) = f^{(\alpha)}(\alpha x).$$

Figure 5.6 shows the graph of the associated cubical multivalued map $F^{(20)}$.

It may seem at first that the approximation given in Figure 5.4 is more accurate than that of Figure 5.6. After all, the sets $F(x)$ are smaller and hence provide more accurate bounds on $f(x)$. However. it must be kept in

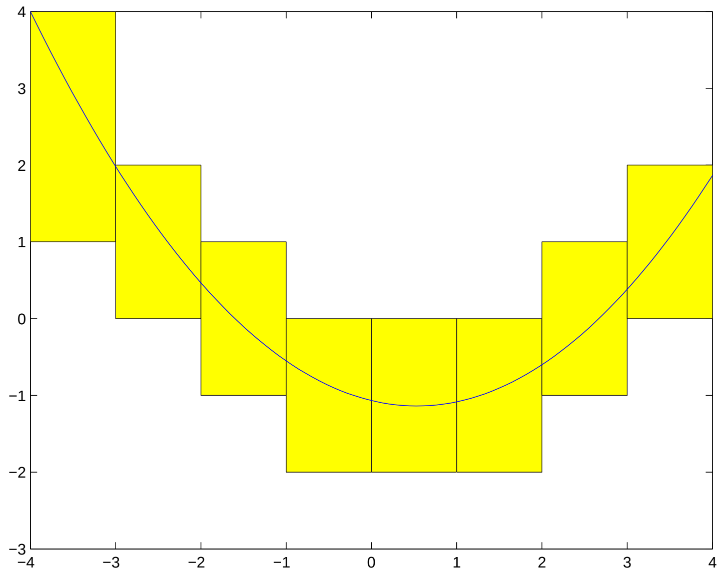


Fig. 5.5. The graph of the cubical multivalued approximation to $f^{(2)}(x) = \left(\frac{x}{2} - \frac{4}{3}\right)\left(\frac{x}{2} + \frac{4}{5}\right)$. The size of the domain has been doubled, but the intervals are all elementary cubes.

mind that we are interested in the images of homology classes rather than particular points. Thus, as will be shown in Chapter 6, modulo some technical conditions, it is sufficient for the homology of the set $F(x)$ to be the same as the homology of a point. In other words, the values of F need to be acyclic. Note that this is satisfied in both cases since $F(x)$ is an interval for any x.

Furthermore, while both Figure 5.4 and Figure 5.6 were obtained by dividing the domain into 80 intervals, in the first the range consisted of approximately 100 intervals while in the latter only 7 interval were used. As one might expect, shorter lists require less calculations, therefore from the computational point of view the latter approximation is preferable.

The reader might ask a question: If so, why don't we just consider the simplest multivalued "approximation" $F : [-2, 2] \rightrightarrows [-3, 4]$ of f given by

$$F(x) := [-3, 4]$$

for all x? It has all the desired properties, its only value is acyclic, and it does not require any calculation at all! We could. However, this is an artifact of the trivial topology of the range $Y = [-3, 4]$. As we shall see shortly, when the homology of the range is nontrivial assuring that the acyclicity condition is satisfied is related to the size of the values.

Exercises

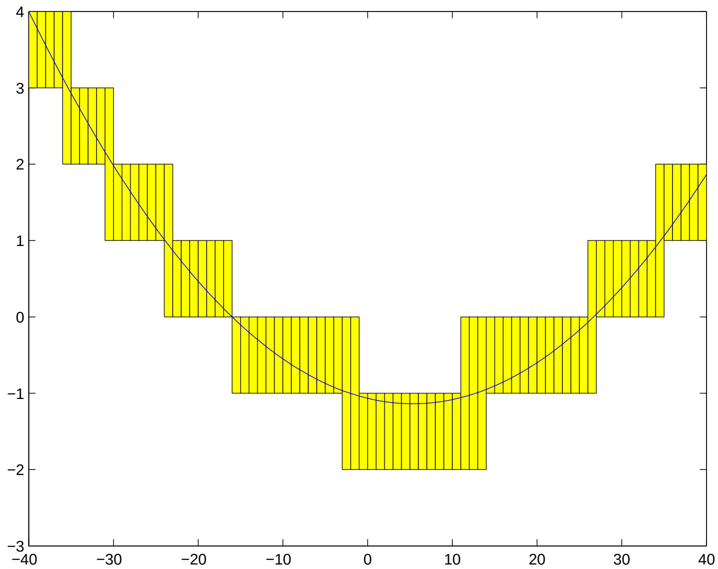


Fig. 5.6. The graph of the cubical multivalued approximation to $f^{(20)}(x)$: $[-40, 40] \to [-3, 4]$.

5.1 Consider the function $f : [-2, 2] \to [-3, 3]$ given by $f(x) = x(x^2 - 1)$.

(a) Use interval arithmetic to construct a cubical multivalued map F that is an outer approximation to f.
(b) Write an explicit formula for $f^{(2)} : [-4, 4] \to [-3, 3]$.
(c) Write a program that draws the graph of a cubical multivalued map $F^{(\alpha)}$ that is an outer approximation for a scaling of f by α.

5.3 Constructing Chain Selectors

In the previous section we considered the problem of approximating maps from one interval to another. Of course the real goal is to use such an approximation to induce a map between homology groups. So in this section we begin with the question: How can we use the information in Figure 5.2 to construct a group homomorphism $f_* : H_*(X) \to H_*(Y)$?

By now we know that maps on homology, are induced by chain maps. Thus, we need to address the following issue. Given a multivalued representation $F : X \rightrightarrows Y$ of $f : X \to Y$, how can we produce an appropriate chain map $\varphi : C(X) \to C(Y)$? Our approach is to use the geometry of F.

Recall that for any $x \in X$ there exists a unique elementary cube $Q \in \mathcal{K}(X)$ such that $x \in \overset{\circ}{Q}$. In analogy to the assumption that $f(x) \in F(x)$ we will require inclusion on the level of the support of images of the chain map, that is,

$$|\varphi(\widehat{Q})| \subset F(\overset{\circ}{Q}) \tag{5.4}$$

Chain maps satisfying this property are called *chain selectors* of F.

To see what this means in practice, let us return to the example of the previous section where F is indicated by Figure 5.2. Recall that the canonical basis of $C_0([-2,2])$ consists of dual vertices

$$\{\widehat{[-2]}, \widehat{[-1]}, \widehat{[0]}, \widehat{[1]}, \widehat{[2]}\}.$$

Similarly, the canonical basis of $C_0(Y)$ consists of

$$\{\widehat{[-2]}, \widehat{[-1]}, \widehat{[0]}, \widehat{[1]}, \widehat{[2]}, \widehat{[3]}, \widehat{[4]}\}.$$

We begin by defining

$$\varphi_0 : C_0(X) \to C_0(Y)$$

which would satisfy the condition 5.4 on vertices. For lack of a better idea let us define

$$\varphi_0(\widehat{[v]}) := \widehat{[\max F(v)]}.$$

For example, $\varphi_0\widehat{[-2]} = \widehat{[4]}$, $\varphi_0\widehat{[-1]} = \widehat{[1]}$, $\varphi_0\widehat{[0]} = \widehat{[0]}$, etc.

We identify the dual vertices in $C_0(X)$ and, respectively, $C_0(Y)$ with the column vectors

$$\mathbf{e}^n_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

of the canonical bases of $\mathbf{Z}^5$, respectively, $\mathbf{Z}^7$, as follows. For the basis of $C_0(X)$,

$$\widehat{[-2]} = \mathbf{e}^5_1,\ \widehat{[-1]} = \mathbf{e}^5_2,\ \widehat{[0]} = \mathbf{e}^5_3,\ \ldots\ \widehat{[2]} = \mathbf{e}^5_5,$$

and for $C_0(Y)$,

$$\widehat{[-2]} = \mathbf{e}^7_1,\ \widehat{[-1]} = \mathbf{e}^7_2,\ \widehat{[0]} = \mathbf{e}^7_3,\ \ldots\ \widehat{[4]} = \mathbf{e}^7_7.$$

The matrix of φ with respect to those bases is

$$\varphi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is natural to ask what would happen if we make a different choice, e.g.

$$\varphi_0(\widehat{[v]}) := [\min \widehat{F(v)}].$$

We could get an entirely different chain map, but it will be proved in the next chapter that both choices lead to the same map in homology. When defining $\varphi_1 : C_1(X) \to C_1(Y)$, on dual edges in $C_1(X)$, we cannot choose an arbitrary dual edge of $C_1(Y)$ as we did it for vertices because we face an additional constraint coming from the condition

$$\partial_1 \varphi_1 = \varphi_0 \partial_1$$

for a chain map. We will attempt to "lift" the definition of φ_0 to dimension one so that this condition is satisfied. Consider the interval $[-2,-1] \subset [-2,2]$. How should we define $\varphi_1(\widehat{[-2,-1]})$? Let us return to the level of topology to find an answer. We know that

$$|\varphi_0(\widehat{[-2]})| = |\widehat{[4]}| = [4] \quad \text{and} \quad |\varphi_0(\widehat{[-1]})| = |\widehat{[1]}| = [1].$$

Since we began with a continuous map and since -2 and -1 are the endpoints of the interval $[-2,-1]$ it seems reasonable that we want the image of $[-2,-1]$ to stretch from the image of -2 to the image of -1, i.e. to go from 4 to 1. Figure 5.7 is an attempt to graphically indicate this idea. The polygonal curve indicated in red joins the pairs of vertices of the form $([v], |\varphi_0(\widehat{[v]})|)$. In particular the projection onto the Y-axis of its first segment is the interval extending from the image of -2 to the image of -1.

Of course this needs to be done algebraically, so let

$$\varphi_1(\widehat{[-2,-1]}) := -\widehat{[1,2]} - \widehat{[2,3]} - \widehat{[3,4]}$$

and observe that

$$|\varphi_1(\widehat{[-2,-1]})| = [1,4] \subset F((-2,-1))$$

so condition 5.4 is satisfied. As in the case of the zero dimensional level, we do not claim that this is a unique choice. Why the minus signs? As we proceed in the direction of the x-axis, i.e. from -2 to -1 the image of the interval goes in the opposite direction of the y-axis, i.e. from 4 to 1.

Similarly, we can set

$$\varphi_1(\widehat{[-1,-0]}) := -\widehat{[0,1]}.$$

But what about $\varphi_1(\widehat{[0,1]})$ where $\varphi_1(\widehat{0}) = \varphi_1(\widehat{1}) = \widehat{0}$? Since the two endpoints are the same, let us just declare that $\varphi_1(\widehat{[0,1]})$ does not map to any intervals, i.e. that $\varphi_1(\widehat{[0,1]}) := 0$.

Again, we want to express φ_1 as a matrix. So let us identify dual edges with canonical basis vectors as it was previously done for vertices. For $C_1([-2,2])$ we put

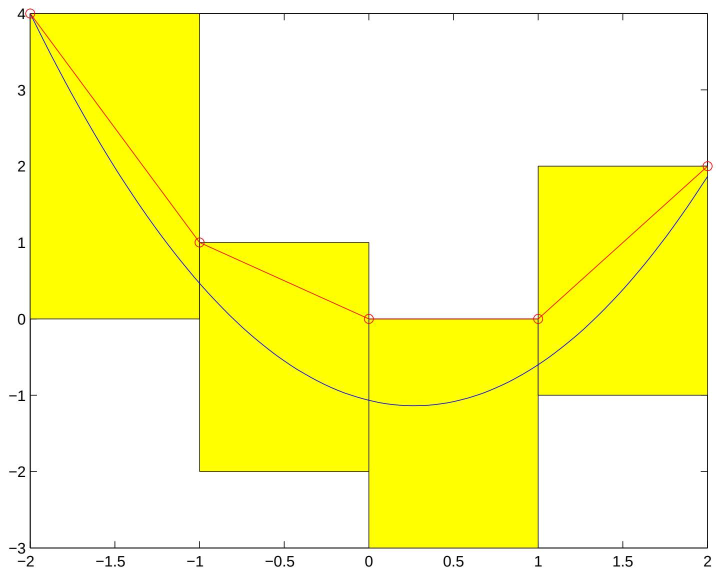


Fig. 5.7. The cubical map F which acts as an outer approximation of f. Given a vertex v, $\varphi_0(\widehat{[v]}) := [\widehat{\max F(v)}]$. The red circles indicate pairs of vertices $([v], |\varphi_0(\widehat{[v]})|)$ while the red segments connecting them can be used to localize, on the Y-axis, the images of the corresponding edges under the chain map φ_1.

$$\widehat{[-2,-1]} = \mathbf{e}_1^4,\ \widehat{[-1,0]} = \mathbf{e}_2^4,\ \widehat{[0,1]} = \mathbf{e}_3^4,\ \widehat{[1,2]} = \mathbf{e}_4^4,$$

and for $C_1([-2,4])$ we put

$$\widehat{[-2,-1]} = \mathbf{e}_1^6,\ \widehat{[-1,0]} = \mathbf{e}_2^6,\ \ldots\ \widehat{[3,4]} = \mathbf{e}_6^4.$$

Applying the reasoning described above to each of the intervals we obtain the following matrix

$$\varphi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The reader should check that φ is a chain map.

Recall that chain maps generate maps on homology, so we define $f_* : H_*(X) \to H_*(Y)$ by

$$f_* := \varphi_*$$

Since X and Y are acyclic we know that

$$H_k(X) \cong H_k(Y) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the only interesting map is

$$f_0 : H_0(X) \to H_0(Y).$$

We know from Section 2.3, that all dual vertices in X are homologous and similarly for Y. So, let us take, for example, the equivalence class $\zeta := \left[\widehat{[-2]}\right]$ of $\widehat{[-2]}$ as a generator of $H_0(X)$ and the equivalence class $\xi := \left[\widehat{[3]}\right]$ of $\widehat{[3]}$ as a generator of $H_0(X)$. Since $\varphi(\widehat{[-2]}) = \widehat{[4]}$ and $\widehat{[4]} \sim \widehat{[3]}$, f_0 is defined on the generator by the formula

$$f_0(\zeta) := \left[\varphi_0(\widehat{[-2]})\right] = \left[\widehat{[4]}\right] = \left[\widehat{[3]}\right] = \xi.$$

This is probably a good place to restate the caveat given at the beginning of the chapter. We are motivating the ideas behind homology maps at this point. If you do not find these definitions and constructions completely rigorous that is good, they are not. We will fill in the details later. For the moment we are just trying to get a feel for how we can obtain group homomorphisms from continuous maps.

Exercises

5.2 Construct another chain selector of F given in this section by starting from the definition

$$\varphi_0(\widehat{[v]}) := [\widehat{\min F(v)}].$$

Give an argument why your chain map induces the same homomorphism in homology as the one discussed in this text.

5.4 Maps of Γ^1

Up to now we have considered maps from one interval to another. Since intervals are acyclic spaces it is not surprising that the resulting maps on homology are not very interesting. So let us consider a space with non-trivial homology such as Γ^1. Unfortunately, it is rather difficult to draw the graph of a function $f : \Gamma^1 \to \Gamma^1$. So we will employ a trick. In order to draw simple pictures we will think of Γ^1 as the interval $[0, 4]$ but identify the endpoints.

More precisely, the identification of a point $x = (x_1, x_2) \in \Gamma^1$ with a number $t \in [0, 4]$ is made via the function $t \mapsto (x_1(t), x_2(t))$ defined as follows.

$$x(t) = (x_1(t), x_2(t)) := \begin{cases} (0,0) + t(1,0) & \text{if } t \in [0,1], \\ (1,0) + (t-1)(0,1) & \text{if } t \in [1,2], \\ (1,1) + (t-2)(-1,0)) & \text{if } t \in [2,3], \\ (0,1) + (t-3)(0,-1) & \text{if } t \in [3,4]. \end{cases} \quad (5.5)$$

Note that the numbers $t \in (0,4)$ correspond bijectively to points $x \in \Gamma^1 \setminus \{(0,0)\}$ but $x(0) = x(4) = (0,0)$, so the identification $t \sim t+4$ makes sense here. Also note that $t = 0,1,2,3$ and $4 \sim 0$ correspond to elementary vertices of Γ^1 while the intervals $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$ correspond to respective elementary edges by going counterclockwise around the square $[0,1]^2$.

In fact, it is useful to go a step further and think of Γ^1 as the real line where one identifies each point t with its translate $t+4$.

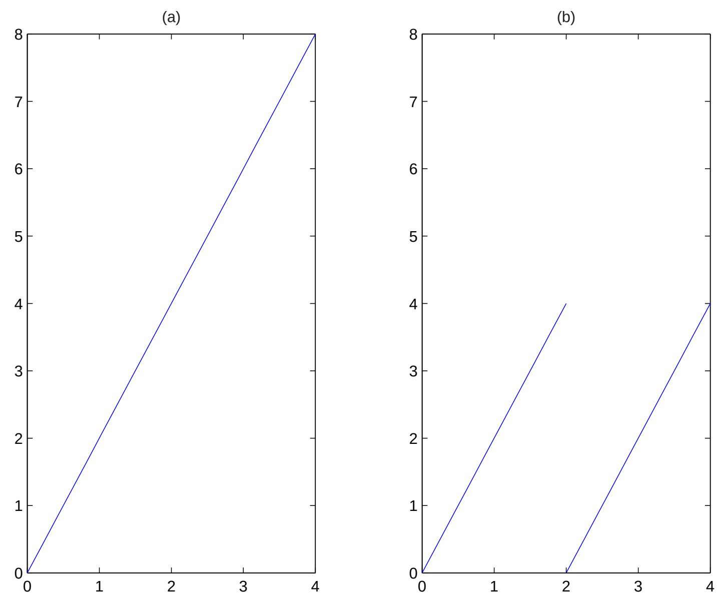


Fig. 5.8. Two versions of the graph of $f(t) = 2t$. The left hand drawing indicates $f : [0,4] \to \mathbf{R}$. In the right hand drawing we have made the identification of $y \sim y+4$ and so can view $f : [0,4] \to [0,4]$. It is important to keep in mind that on both the x and y axis we make the identification of $0 \sim 4 \sim 8$. Thus $f(0) \sim 0 \sim f(2) \sim 4 \sim f(4) \sim 8$.

To see how this works in practice consider the function $f : [0,4] \to \mathbf{R}$ given by $f(t) = 2t$. We want to think of f as a map from $\Gamma^1 \to \Gamma^1$ and do this via the identification of $t \sim t+4$ (see Figure 5.8).

While this process allows us to draw nice figures it must be kept in mind that what we are really interested in f as a continuous mapping from Γ^1 to Γ^1. How should we interpret the drawing in Figure 5.8(b)? Observe that as we move across the interval $[0,2]$ the graph of f covers all of $[0,4]$. So going half way around Γ^1 in the domain corresponds to going once around Γ^1 in the image. Thus, going all the way around Γ^1 in the domain results in going twice around Γ^1 in the image. In other words, f wraps Γ^1 around itself twice. In

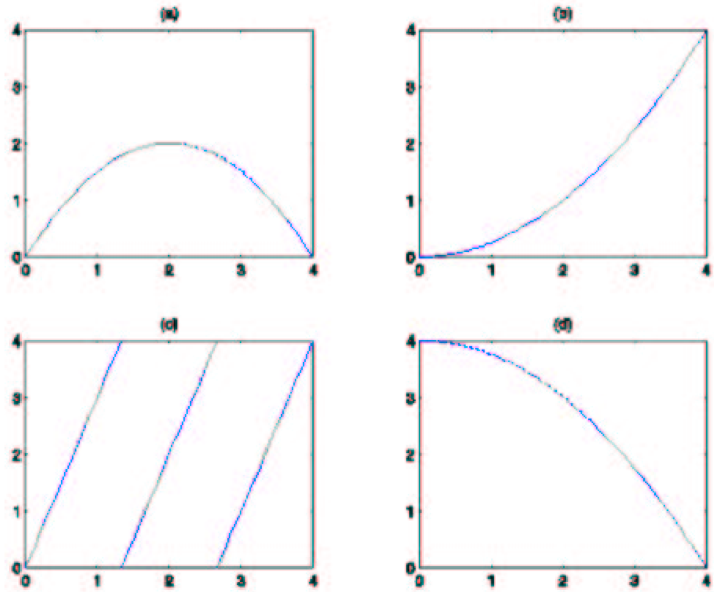


Fig. 5.9. Four different maps $f : \Gamma^1 \to \Gamma^1$. How do these different f's wrap Γ^1 around Γ^1? (a) $f(t) = \frac{1}{2}t(4-t)$ wraps the interval $[0,2]$ half way around Γ^1 and then over the interval $[2,4]$ f unwraps it. Thus, we could say that the total amount of wrapping is 0. (b) $f(t) = \frac{1}{4}t^2$ wraps Γ^1 once around Γ^1. (c) $f(t) = 3t$ wraps Γ^1 three times around Γ^1. (d) $f(t) = 4 - \frac{1}{4}t^2$ wraps Γ^1 once around Γ^1, but in the opposite direction from the example in (b).

Figure 5.9 we show a variety of different maps and indicate how many times they wrap Γ^1 around itself. Our goal in this section is to see if we can detect the differences in these maps algebraically.

Recall that

$$H_k(\Gamma^1) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

We will focus our attention on $f_1 : H_1(\Gamma^1) \to H_1(\Gamma^1)$.

Since $H_1(\Gamma^1) \cong \mathbf{Z}$, it is generated by one element $\zeta := [z]$ where z is a counter-clockwise cycle expressed, under the discussed identification of Γ^1 with $[0,4]$ as

$$z := \widehat{[0,1]} + \widehat{[1,2]} + \widehat{[2,3]} + \widehat{[3,4]}.$$

If the construction of F and its chain selector φ for a map $f : \Gamma^1 \to \Gamma^1$ does not require rescaling, then it is enough to determine the value of φ_1 on z. What happens if we do need a rescaling, say, by $\alpha = 2$?. Formally, the rescaling by 2 of $\Gamma^1 \subset \mathbf{R}$ should be defined as a map $\Lambda^{(2,2)} : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\Lambda^{(2,2)}(x_1, x_2) := (2x_1, 2x_2).$$

Thus the image of $X := \Gamma^1$ under this rescaling formally is the boundary of the square $[0,2]^2$. This will be the way we will develop the formal definitions in the next chapter. However, we will content ourselves by now with the simpler presentation of X as the interval $[0,4]$ with identified endpoints. An analogy of that presentation for the rescaled set $X^{(2)}$ is viewing it as the interval $X^{(2)} = [0,8]$ with identified endpoints. In other words, we consider the rescaling $\Lambda^{(2)}(t) = 2t$ in $\mathbf{R}$, as in the first section. The reader is encouraged to check that the homology of $X^{(2)}$ is the same as that of Γ^1, in particular, $H_1(X^{(2)}) \cong \mathbf{Z}$. Moreover this homology group is generated by one element $\zeta^{(2)} := [z^{(2)}]$ where $z^{(2)}$ is a counter-clockwise cycle expressed, under the discussed identification with $[0,8]$, as

$$z^{(2)} := \widehat{[0,1]} + \widehat{[1,2]} + \widehat{[2,3]} + \widehat{[3,4]} + \widehat{[4,5]} + \widehat{[5,6]} + \widehat{[6,7]} + \widehat{[7,8]}.$$

If the map f requires rescaling by $\alpha = 2$, we will produce a map $f_1^{(2)} : H_1(X^{(2)}) \to H_1(X)$. Our goal is to get the map $f_1 : H_1(X) \to H_1(X)$, so we will need to compose $f_1^{(2)}$ with the homology map $\Lambda_1 : H_1(X) \to H_1(X^{(2)})$ of our rescaling. Observe that $\Lambda^{(2)}$ sends the support of z to the support of $z^{(2)}$ and it preserves orientation on the contour, so it can be guessed that we should have the formula

$$\Lambda_1^{(2)}(\zeta) = \zeta^{(2)}.$$

However, we have to omit the verification at this time.

After this preliminary discussion, we are now ready to show the construction of homology map for some specific maps f.

Example 5.5 Let us begin by considering the map $f : \Gamma^1 \to \Gamma^1$ given by

$$f(t) = \frac{1}{2}t(4-t)$$

which is drawn in Figure 5.9(a). The first step is to obtain an approximation for f which is done using the interval approximation $\bar{f}$ defined by (5.1). In Figure 5.10 we indicate the resulting cubical multivalued map F that is an outer approximation for f.

Recall that we defined the images of vertices via equation (5.3). Thus,

$$\begin{aligned} F([1]) &= F([0,1]) \cap F([1,2]) \\ &= [1,3] \cup [4]. \end{aligned}$$

This is troubling. What we are saying is that using this procedure the outer approximation of a point is the union of two disjoint sets. This, in particular is not an acyclic set which was what we wanted to obtain and thus we turn to the scaling technique introduced at the end of Section 5.2.

To avoid this problem we can make use of the scaling $\alpha = 2$ to Γ^1. Recall that the scaling is only applied to the domain. To keep track of this let us write $f : X \to Y$ where $X = Y = \Gamma^1$. Then under the scaling $\Lambda^{(\alpha)} : \mathbf{R} \to \mathbf{R}$,

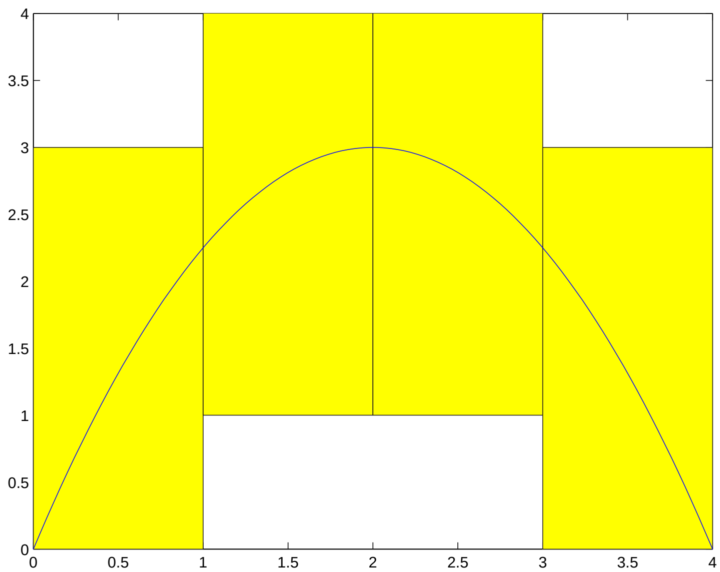


Fig. 5.10. The outer approximation for the map $f(t) = \frac{1}{2}t(4-t)$ Because we are identifying 0 and 4, $F(1) = F(3) = [1,3] \cup [4]$. Thus for this outer approximation the image of a point need not be an acyclic set.

$X^{(\alpha)} := \Lambda^{(\alpha)}(X)$. Note that $X^{(\alpha)}$ can be viewed as the interval $[0,8]$ in $\mathbf{R}$ with 0 and 8 identified. Since the range is not being changed, we hold on to viewing Y as the interval $[0,4]$ with the identification of 0 with 4. Performing the same computations as before we can compute an outer approximation for $f^{(2)}$ and obtain result indicated in Figure 5.11.

Using the same rules as before we end up with the multivalued map

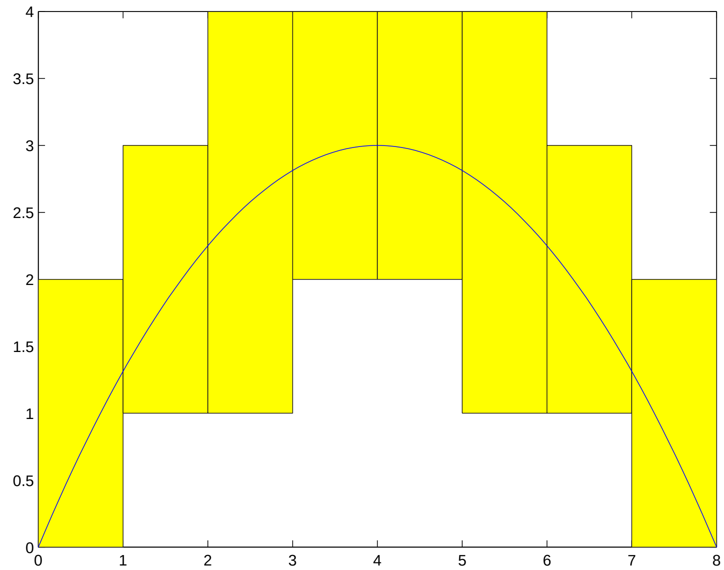


Fig. 5.11. The outer approximation for the map $f^{(2)}(t) = \frac{1}{4}t(4 - \frac{t}{2})$.

$$
F^{(2)}(t) = \begin{cases}
[0,2] \cup [3,4] & \text{if } t = 0 \\
[0,2] \cup [3,4] & \text{if } t \in (0,1) \\
[0,2] & \text{if } t = 1 \\
[0,2] & \text{if } t \in (1,2) \\
[1,2] & \text{if } t = 2 \\
[1,3] & \text{if } t \in (2,3) \\
[1,3] & \text{if } t = 3 \\
[1,3] & \text{if } t \in (3,4) \\
[1,3] & \text{if } t = 4 \\
[1,3] & \text{if } t \in (4,5) \\
[1,3] & \text{if } t = 5 \\
[1,3] & \text{if } t \in (5,6) \\
[1,2] & \text{if } t = 6 \\
[0,2] & \text{if } t \in (6,7) \\
[0,2] & \text{if } t = 7 \\
[0,2] \cup [3,4] & \text{if } t \in (7,8) \\
[0,2] \cup [3,4] & \text{if } t = 8
\end{cases}
$$

We shall skip the verification that all the values are now acyclic. Since $f_1^{(2)}$ is to be a linear map, it will be enough to define its value on the generator $\zeta^{(2)}$ of the homology group $H_1(X^(2))$.

Having determined $F^{(2)}$ we will construct its chain selector $\varphi_0 : C_0(X^{(2)}) \to C_0(X)$ in the same manner as in Section 5.3. Set $\varphi_0(v) = \max F^{(2)}(v)$ for any vertex v. Thus for example, $\varphi_0(\widehat{[0]}) = \widehat{[4]}$, $\varphi_0(\widehat{[1]}) = \widehat{[2]}$, etc. Having defined φ_0, the construction of $\varphi_1 : C_1(X^{(2)}) \to C_1(X)$ also follows as in Section 5.3. Of course, because of the identification, it is a little more subtle in this case. In particular, $\varphi_0(\widehat{[0]}) = \widehat{[4]} = \widehat{[0]}$.

Again, we assign canonical vectors to the duals of elementary intervals as follows. In $C_1(X^{(2)})$,

$$\widehat{[0,1]} = \mathbf{e}_1^8,\ \widehat{[1,2]} = \mathbf{e}_2^8,\ \ldots \widehat{[7,8]} = \mathbf{e}_8^8,$$

and in $C_1(Y)$,

$$\widehat{[0,1]} = \mathbf{e}_1^4,\ \widehat{[1,2]} = \mathbf{e}_2^4,\ \widehat{[2,3]} = \mathbf{e}_3^4,\ \widehat{[2,3]} = \mathbf{e}_4^4.$$

Using these bases we can write

$$\varphi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As before, the negative signs arise because the image of the interval goes in the opposite direction from the ordering of the y-axis.

In order to understand the induced map from $H_1(X^{(2)})$ to $H_1(X)$ we need to see how φ_1 acts on $z^{(2)}$.

In vector notation, we have

$$z^{(2)} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

It is instantly checked that $\varphi_1(z^{(2)}) = 0$. Hence $f_1^{(2)} = 0$ and, consequently, $f_1 : H_1(X^{(2)}) \to H_1(\Gamma^1)$ is the zero homomorphism too. This is interpreted by observing that the winding does not occur here.

Example 5.6 Let's repeat this calculation for the map

$$f(t) = \frac{1}{4}t^2.$$

We proceed exactly as before using the interval map $\bar{f}$ defined by (5.1). Figure 5.12 shows the resulting multivalued map and indicates how the chain maps[1] are defined. Of course for purposes of calculation we want to view the chain map as a matrix

$$\varphi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

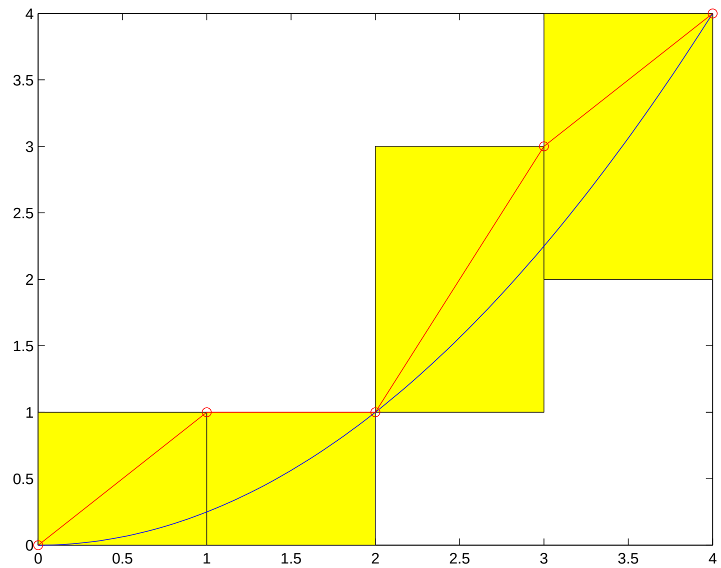


Fig. 5.12. The outer approximation for the map $f(t) = \frac{1}{4}t^2$. The red circles indicate pairs of vertices $([v], |\varphi_0(\widehat{[v]})|)$ while the red segments connecting them can be used to localize, on the Y-axis, the images of the corresponding edges under the chain map φ_1.

To determine the image of z under φ_1 we evaluate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, $f_1 : H_1(X) \to H_1(X)$ is given by $f_1(\zeta) = \zeta$. Observe that this is corresponds to the fact that $f(t) = t^2$ wraps Γ^1 around itself once in the clockwise direction.

Example 5.7 We shall do one more example, that of

$$f(t) = 2t.$$

Figure 5.13 shows the multivalued map that acts as an outer approximation when a scaling $\alpha = 2$ is used.

The rescaling of $[0, 4]$ by 3 gives the interval $[0, 12]$ which is composed of 12 elementary intervals. Following exactly the same process as in the previous cases we obtain

[1] The reader is encouraged to define those chain maps explicitly.

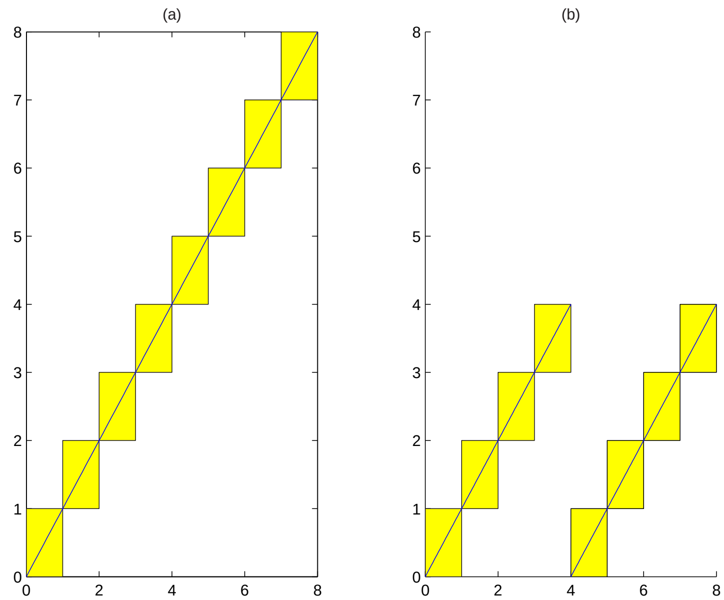


Fig. 5.13. The outer approximation for the rescaling by 2 of the map $f(t) = 2t$. (a) The graph of the multivalued map with domain $[0, 4]$ and image $[0, 8]$. (b) The same graph however the interval $[0, 4]$ has been identified with the interval $[4, 8]$.

$$\varphi_1 = \begin{bmatrix} 1\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0 \end{bmatrix}$$

Again viewing how this acts on the generator $\zeta^{(3)}$ of $H_1(X^{(3)})$ (the construction of this generator, by analogy to previous cases, is left to the reader) we have

$$\varphi_1(z^{(3)}) = \begin{bmatrix} 1\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = 2z$$

Finally, we use the homology of rescaling to get

$$f_1(\zeta) = f_1^{(3)} \Lambda_1^{(3)}(\zeta) = f_1^{(3)}(\zeta^{(3)}) = 2\zeta.$$

Thus the homology map on the first level is multiplication by 2. Observe that again it corresponds to the fact that $f(t) = 2t$ winds Γ^1 around itself twice.

Exercises

5.3 Compute $f_1 : H_1(\Gamma^1) \to H_1(\Gamma^1)$ for $f(t) = -t$.

5.4 To motivate the use of computers try to compute $f_1 : H_1(\Gamma^1) \to H_1(\Gamma^1)$ for $f(t) = 3t$.

6

Homology of Maps

6.1 Representable Sets

Using elementary cells we can define a larger class of topological spaces.

Definition 6.1 A set $Y \subset \mathbf{R}^d$ is *representable* if it is a finite union of elementary cells. The family of representable sets in $\mathbf{R}^d$ is denoted by $\mathcal{R}^d$.

As an immediate consequence of Proposition 2.15(v) we get

Proposition 6.2 *Every elementary cube is representable.*

From Proposition 2.15(iii) we get

Proposition 6.3 *If $A, B \subset \mathbf{R}^d$ are representable then $A \cup B$, $A \cap B$ and $A \setminus B$ are representable.*

We also have the following proposition

Proposition 6.4 *$X \subset \mathbf{R}^d$ is a cubical set if and only if it is closed and representable.*

Proof. Assume X is a cubical set. Then as a finite union of elementary cubes it is closed and by Proposition 6.2 and Proposition 6.3 it is representable. Thus assume that X is closed and representable. Then

$$X = \overset{\circ}{Q}_1 \cup \overset{\circ}{Q}_2 \cup \ldots \cup \overset{\circ}{Q}_m$$

for some $Q_i \in \mathcal{K}^d$ and by Proposition 2.15(iv)

$$X = \operatorname{cl} X = \operatorname{cl} \overset{\circ}{Q}_1 \cup \operatorname{cl} \overset{\circ}{Q}_2 \cup \ldots \cup \operatorname{cl} \overset{\circ}{Q}_m = Q_1 \cup Q_2 \cup \ldots Q_m,$$

which shows that X is cubical. $\square$

Proposition 6.5 *The set $A \subset \mathbf{R}^d$ is representable if and only if the following two conditions are satisfied* *(i)* $\operatorname{cl} A$ *is bounded.*

(ii) for every $Q \in \mathcal{K}^d$ $\overset{\circ}{Q} \cap A \neq \emptyset$ *implies* $\overset{\circ}{Q} \subset A$.

Proof. Assume A is representable. Then $A = \overset{\circ}{Q}_1 \cup \overset{\circ}{Q}_2 \cup \ldots \cup \overset{\circ}{Q}_m$ for some $Q_i \in \mathcal{K}^d$ and by Proposition 2.15(iv)

$$\operatorname{cl} A = \operatorname{cl} \overset{\circ}{Q}_1 \cup \operatorname{cl} \overset{\circ}{Q}_2 \cup \ldots \cup \operatorname{cl} \overset{\circ}{Q}_m = Q_1 \cup Q_2 \cup \ldots \cup Q_m$$

is bounded. Also, if $\overset{\circ}{Q} \cap A \neq \emptyset$ for some $Q \in \mathcal{K}^d$ then $\overset{\circ}{Q} \cap \overset{\circ}{Q}_i \neq \emptyset$ for some $i \in \{1, 2, \ldots, m\}$. It follows from Proposition 2.15(iii) that $Q = Q_i$ and consequently $\overset{\circ}{Q} \subset A$. Hence (i) and (ii) are satisfied.

On the other hand, if properties (i) and (ii) are satisfied then by Proposition 2.15(i)

$$A = \bigcup \{A \cap \overset{\circ}{Q} \mid Q \in \mathcal{K}\} = \bigcup \{\overset{\circ}{Q} \mid Q \in \mathcal{K},\ \overset{\circ}{Q} \subset A\}. \tag{6.1}$$

Since $\operatorname{cl} A$ is bounded, it follows from Proposition 2.15(ii) that the last union in (6.1) is finite. Hence A is a representable set. □

Proposition 6.6 *Assume* $A \in \mathcal{R}^d$. *Then* A *is closed if and only if for every* $Q \in \mathcal{K}^d$

$$\overset{\circ}{Q} \subset A \;\Rightarrow\; Q \subset A. \tag{6.2}$$

Similarly, A *is open if and only if for every* $Q \in \mathcal{K}^d$

$$\overset{\circ}{Q} \cap A = \emptyset \;\Rightarrow\; Q \cap A = \emptyset. \tag{6.3}$$

Proof. If A is closed and $\overset{\circ}{Q} \subset A$ for some $Q \in \mathcal{K}$, then by Proposition 2.15(iv) $Q = \operatorname{cl} \overset{\circ}{Q} \subset A$. To prove that this condition is sufficient take the decomposition $A = \overset{\circ}{Q}_1 \cup \overset{\circ}{Q}_2 \cup \ldots \cup \overset{\circ}{Q}_m$ of A into open elementary cubes. We have

$$\operatorname{cl} A = \operatorname{cl} \overset{\circ}{Q}_1 \cup \operatorname{cl} \overset{\circ}{Q}_2 \cup \ldots \cup \operatorname{cl} \overset{\circ}{Q}_m = Q_1 \cup Q_2 \cup \ldots \cup Q_m \subset A,$$

which proves that A is closed.

Assume in turn that A is open. Then $\overset{\circ}{Q} \cap A = \emptyset$ implies $\operatorname{cl}(\overset{\circ}{Q}) \cap A = \emptyset$ and by Proposition 2.15(iv) we get $Q \cap A = \emptyset$.

To show that property (6.3) is sufficient we will prove that it implies that $R \setminus A$ is closed. For this end take a $Q \in \mathcal{K}^d$ such that $\overset{\circ}{Q} \subset R \setminus A$. Then $\overset{\circ}{Q} \cap A = \emptyset$ and by (6.3) $Q \cap A = \emptyset$, i.e. $Q \subset R \setminus A$. Thus $R \setminus A$ is closed, i.e. A is open in R. But $A \subset \operatorname{int} R$, hence A is open. □

Definition 6.7 Let $A \subset \mathbf{R}^d$ be a bounded set. Then the *open hull* of A is

$$\operatorname{oh}(A) := \bigcup \{ \overset{\circ}{Q} \mid Q \in \mathcal{K}, Q \cap A \neq \emptyset \}, \tag{6.4}$$

and the *closed hull* of A is

$$\operatorname{ch}(A) := \bigcup \{ Q \mid Q \in \mathcal{K}, \overset{\circ}{Q} \cap A \neq \emptyset \}. \tag{6.5}$$

Example 6.8 Consider the vertex $P = [0] \times [0] \in \mathbf{R}^2$. Then,

$$\operatorname{oh}(P) = \{ (x_1, x_2) \in \mathbf{R}^2 \mid -1 < x_i < 1 \}.$$

Generalizing this example leads to the following result.

Proposition 6.9 *Let $P = [a_1] \times \cdots \times [a_d] \in \mathbf{R}^d$ be an elementary vertex. Then,*

$$\operatorname{oh}(P) = (a_1 - 1, a_1 + 1) \times \cdots \times (a_d - 1, a_d + 1).$$

The names chosen for $\operatorname{oh}(A)$ and $\operatorname{ch}(A)$ are justified by the following proposition.

Proposition 6.10 *Assume $A \subset \mathbf{R}^d$. Then*

(i) $A \subset \operatorname{oh}(A)$ *and* $A \subset \operatorname{ch}(A)$.
(ii) *The set* $\operatorname{oh}(A)$ *is open and representable.*
(iii) *The set* $\operatorname{ch}(A)$ *is closed and representable.*
(iv) $\operatorname{oh}(A) = \bigcap \{ U \in \mathcal{R}^d \mid U \text{ is open and } A \subset U \}$
(v) $\operatorname{ch}(A) = \bigcap \{ B \in \mathcal{R}^d \mid B \text{ is closed and } A \subset B \}$. *In particular, if K is a cubical set such that $A \subset K$, then* $\operatorname{ch}(A) \subset K$.
(vi) $\operatorname{oh}(\operatorname{oh}(A)) = \operatorname{oh}(A)$ *and* $\operatorname{ch}(\operatorname{ch}(A)) = \operatorname{ch}(A)$.
(vii) *If $y \in \operatorname{oh}(x)$, then* $\operatorname{ch}(x) \subset \operatorname{ch}(y)$.
(viii) $Q \in \mathcal{K}^d$ *and* $x \in \overset{\circ}{Q}$ *implies that* $\operatorname{ch}(x) = Q$.
(ix) *Let $Q \in \mathcal{K}^d$ and let $x, y \in \overset{\circ}{Q}$. Then,* $\operatorname{oh}(x) = \operatorname{oh}(y)$ *and* $\operatorname{ch}(x) = \operatorname{ch}(y)$.

Proof. (i) Let $x \in A$. By Proposition 2.15(i) there exists an elementary cube Q such that $x \in \overset{\circ}{Q}$. It follows that $\overset{\circ}{Q} \cap A \neq \emptyset \neq Q \cap A$. Hence $x \in \operatorname{oh}(A)$ and $x \in \operatorname{ch}(A)$.

(ii) By Proposition 2.15(ii) the union in (6.4) is finite. Therefore the set $\operatorname{oh}(A)$ is representable. To prove that $\operatorname{oh}(A)$ is open we will show that it satisfies (6.3). Let $P \in \mathcal{K}^d$ be such that $\overset{\circ}{P} \cap \operatorname{oh}(A) = \emptyset$. Assume that $P \cap \operatorname{oh}(A) \neq \emptyset$. Then there exists a $Q \in \mathcal{K}$ such that $Q \cap A \neq \emptyset$ and $P \cap \overset{\circ}{Q} \neq \emptyset$. Since P is representable, it follows from Proposition 6.5 that $\overset{\circ}{Q} \subset P$. Therefore $Q = \operatorname{cl} \overset{\circ}{Q} \subset P$, i.e. $P \cap A \neq \emptyset$. This means that $\overset{\circ}{P} \subset \operatorname{oh}(A)$, a contradiction. It follows that $\operatorname{oh}(A)$ is open.

(iii) The set $\mathrm{ch}\,(A)$ is closed since it is the finite union of closed sets. By Proposition 6.2 $\mathrm{ch}\,(A)$ is representable.

(iv) Observe that since $\mathrm{oh}\,(A)$ is open, representable and contains A,

$$\bigcap\{U \in \mathcal{R}^d \mid U \text{ is open and } A \subset U\} \subset \mathrm{oh}\,(A).$$

To show the opposite inclusion take an open set $U \in \mathcal{R}^d$ such that $A \subset U$. Let $x \in \mathrm{oh}\,(A)$. Then there exists a $Q \in \mathcal{K}$ such that $A \cap Q \neq \emptyset$ and $x \in \overset{\circ}{Q}$. It follows that $\emptyset \neq Q \cap U = \mathrm{cl}\ \overset{\circ}{Q} \cap U$, i.e. $\overset{\circ}{Q} \cap U \neq \emptyset$. By Proposition 6.5 $\overset{\circ}{Q} \subset U$, hence $x \in U$. This shows that $\mathrm{oh}\,(A) \subset U$ and since U is arbirtary,

$$\mathrm{oh}\,(A) \subset \bigcap\{U \in \mathcal{R}^d \mid U \text{ is open and } A \subset U\}.$$

(v) Since $\mathrm{ch}\,(A)$ is closed, representable and contains A,

$$\bigcap\{B \in \mathcal{R}^d \mid B \text{ is closed and } A \subset B\} \subset \mathrm{ch}\,(A).$$

Let $K \in \mathcal{R}^d$ be a closed set which contains A. We will show that $\mathrm{ch}\,(A) \subset K$. For this end take an $x \in \mathrm{ch}\,(A)$. Then there exists a $Q \in \mathcal{K}$ such that $\overset{\circ}{Q} \cap A \neq \emptyset$ and $x \in Q$. It follows that $\overset{\circ}{Q} \cap K \neq \emptyset$ and consequently $\overset{\circ}{Q} \subset K$. Hence $Q \subset K$ and $x \in K$. This shows that $\mathrm{ch}\,(A) \subset K$ and since K is arbitrary,

$$\mathrm{ch}\,(A) \subset \bigcap\{B \in \mathcal{R}^d \mid B \text{ is closed and } A \subset B\}.$$

(vi) This follows immediately from (iv) and (v).

(vii) Observe that since $y \in \mathrm{oh}\,(x)$, there exists a $P \in \mathcal{K}$ such that $y \in \overset{\circ}{P}$ and $x \in P$. Take a $z \in \mathrm{ch}\,(x)$. Then there exists a $Q \in \mathcal{K}$ such that $z \in Q$ and $x \in \overset{\circ}{Q}$. It follows that $\overset{\circ}{Q} \subset P$, hence also $Q \subset P$ and consequently $z \in P$, which proves (vii).

(viii) This is straightforward.

(ix) Let $z \in \mathrm{oh}\,(x)$. Then there exists a $Q \in \mathcal{K}$ such that $z \in \overset{\circ}{Q}$ and $x \in Q$. It follows that $\overset{\circ}{P} \subset Q$, i.e. $y \in Q$. Consequently $z \in \mathrm{oh}\,(y)$ and $\mathrm{oh}\,(x) \subset \mathrm{oh}\,(y)$. The same way one proves that $\mathrm{oh}\,(y) \subset \mathrm{oh}\,(x)$. The equality $\mathrm{ch}\,(x) = \mathrm{ch}\,(y)$ follows from (viii).

□

Proposition 6.11 *Assume $x, y \in \mathbf{R}$, $x \leq y$ are two arbitrary real numbers. Then*

$$\mathrm{ch}\,([x, y]) = [\mathrm{floor}\,(x), \mathrm{ceil}\,(x)].$$

Proof. Put $p := \mathrm{floor}\,(x)$ and $q := \mathrm{ceil}\,(x)$. Obviously $[p, q]$ is a cubical set and $[x, y] \subset [p, q]$. Therefore $\mathrm{ch}\,([x, y]) \subset [p, q]$. To show the opposite inclusion take $z \in [p, q]$. If $z \in [x, y]$, the conclusion is obvious. Otherwise

$$z \in [p, x) \cup (y, q] \subset [p, p+1] \cup [q-1, q] \subset \mathrm{ch}\,([x, y]),$$

because $(p, p+1) \cap [x, y] \neq \emptyset \neq (q-1, q) \cap [x, y]$. □

Proposition 6.12 *Let $\Delta_1, \Delta_2, \ldots, \Delta_d$ be bounded intervals. Then*

$$\mathrm{ch}\,(\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d) = \mathrm{ch}\,(\Delta_1) \times \mathrm{ch}\,(\Delta_2) \times \ldots \times \mathrm{ch}\,(\Delta_d).$$

Proof. Since $\mathrm{ch}\,(\Delta_1) \times \mathrm{ch}\,(\Delta_2) \times \ldots \times \mathrm{ch}\,(\Delta_d)$ is a closed representable set, which contains $\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d$ it must also contain $\mathrm{ch}\,(\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d)$. To show the opposite inclusion, take $y := (y_1, y_2, \ldots, y_d) \in \mathrm{ch}\,(\Delta_1) \times \mathrm{ch}\,(\Delta_2) \times \ldots \times \mathrm{ch}\,(\Delta_d)$. Then there exist elementary intervals I_i such that $y_i \in I_i$ and $\overset{\circ}{I}_i \cap \Delta_i \neq \emptyset$ for $i \in \{1, 2, \ldots, n\}$. Hence $\overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d$ is an elementary cell, which has a non-empty intersection with $\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d$ and $y \in I_1 \times I_2 \times \ldots \times I_d$. Therefore $y \in \mathrm{ch}\,(\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d)$. □

As an immediate corollary of the last two propositions we obtain the following corollary.

Corollary 6.13 *Let $\Delta_1, \Delta_2, \ldots, \Delta_d$ be bounded intervals. Then* $\mathrm{ch}\,(\Delta_1 \times \Delta_2 \times \ldots \times \Delta_d)$ *is a rectangle.*

Exercises

6.1 Write the input file for the elementary cell $A = (2, 3) \subset \mathbf{R}$ for the program CubTop . Then alternate the functions `closedhull` and `opendhull` to see the representable sets $\mathrm{ch}\,(A)$, $\mathrm{oh}\,(\mathrm{ch}\,(A))$, $\mathrm{ch}\,(\mathrm{oh}\,(\mathrm{ch}\,(A)))$, and so on. This experiment should help you to answer the question in Exercise 6.4.

6.2 The file exR2d.txt in the folder CubTop/Examples contains a representable set A. Run CubTop on the file exR2d.txt with the option -g to see how does it look like. Run CubTop `closedhull` to see $\mathrm{ch}\,(A)$. Then run CubTop `opendhull` on the files of A and of $\mathrm{ch}\,(A)$ to see the effect.

6.3 Given a cubical set $X \subset \mathbf{R}^d$ and a positive integer r, define the *closed cubical ball* of radius r about X by the formula

$$\bar{B}_r(X) := \{y \in R^d \mid \mathrm{dist}\,(y, X) \leq r\},$$

where

$$\mathrm{dist}\,(y, X) := \sup\{||y - x||_0 \mid x \in X\}$$

and $||y - x||_0$ is the supremum norm of $y - x$ (see the formula (12.3) in Chapter 12). Give, with a proof, a formula for $\bar{B}_r(X)$ in terms of closed hulls of X. This will show, in particular, that $\bar{B}_r(X)$ is a cubical set.

6.4 Given a cubical set $X \subset R^d$ and a positive integer r, define the *open cubical ball* about X by the formula

$$B_r(X) := \{y \in R^d \mid \operatorname{dist}(y, X) < r\}.$$

Give, with a proof, a formula for $B_r(X)$ in terms of open and closed hulls of X (you will need them both). This will show, in particular, that $B_r(X)$ is a representable set. What happens if we extend the definition to open balls about bounded, not necessarily closed sets? You may use the experiments in Exercises 6.1 and 6.2 to make the right guess.

6.2 Cubical Multivalued Maps

In the last section we discussed chain maps and the maps they induce on homology. We did not however, discuss how one goes from a continuous map to a chain map. There are a variety of possibilities, each with its advantages and disadvantages. The approach we will adopt involves using multivalued maps to approximate the continuous map. The motivation for this was given in Chapter 5. We now want to formalize these ideas.

Let X and Y be cubical sets. A *multivalued map* $F : X \rightrightarrows Y$ from X to Y is a function from X to subsets of Y, i.e. for every $x \in X$, $F(x) \subset Y$. This class is very broad and since we are interested in computations, we recall Definition 5.3 of a cubical multivalued map already introduced in Chapter 5.

Let X and Y be cubical sets. A multivalued map $F : X \rightrightarrows Y$ is *cubical* if:

1. For every $x \in X$, $F(x)$ is a cubical set.
2. For every $Q \in \mathcal{K}(X)$, $F|_{\overset{\circ}{Q}}$ is constant, i.e. if $x, x' \in \overset{\circ}{Q}$, then $F(x) = F(x')$.

Cubical multivalued maps have the nice feature that on one hand they are maps between topological spaces and may be used to enclose single valued continuous maps (which in general have infinitely many values), on the other hand they may be easily encoded with a finite amount of information which makes them convenient to handle on a computer.

With our goal of using multivalued maps to enclose the image of a continuous function there is a canonical choice of constructing cubical maps. To make this clear let us return to the discussion in Section 5.2 where the use of multivalued maps was first presented. We considered the function $f(x) = (x - \sqrt{2})(x + 1)$ as a map from $X = [-2, 2] \subset \mathbf{R}$ to $Y = [-2, 4] \subset \mathbf{R}$. Using a Taylor representation we derived bounds on f that applied to each $Q \in \mathcal{K}_1(X)$ (see Table 5.1 and Figure 5.2) and extended these bounds to vertices by intersecting the bounds of the adjacent edges.

This approach may be easily generalized in the following way. Given a combinatorial multivalued map $\mathcal{F} : \mathcal{K}_{\max}(X) \rightrightarrows \mathcal{K}(Y)$ (see Definition 5.4) we define the multivalued maps $\lfloor \mathcal{F} \rfloor, \lceil \mathcal{F} \rceil : X \rightrightarrows Y$ by

$$\lfloor \mathcal{F} \rfloor(x) := \bigcap \{|\mathcal{F}(Q)| \mid x \in Q \in \mathcal{K}_{\max}(X)\}\,, \tag{6.6}$$

$$\lceil \mathcal{F} \rceil(x) := \bigcup \{|\mathcal{F}(Q)| \mid x \in Q \in \mathcal{K}_{\max}(X)\}\,, \tag{6.7}$$

where $|\mathcal{F}(Q)|$ is defined by (2.10).

Theorem 6.14 *The multivalued maps $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are cubical.*

Proof. The first property of Definition 5.3 follows from the straightforward fact that the union and intersection of cubical sets is a cubical set. To prove the other observe that by Proposition 2.15(vi) the union in (6.6) and intersection in (6.6) is over the same family of cubes. □

Example 6.15 Let $X = [0,2]$ and let $Y = [-5,5]$. Define $F : X \rightrightarrows Y$ by

$$F(x) := \begin{cases} [-5] & \text{if } x = 0 \\ [-4,-1] & \text{if } x \in (0,1) \\ [0] & \text{if } x = 1 \\ [1,4] & \text{if } x \in (1,2) \\ [5] & \text{if } x = 2 \end{cases}$$

The graph of this function is given in Figure 6.1. Observe that F is a cubical map. However, from several points of view this is not satisfactory for our needs. The first is that intuitively it should be clear that a map of this type cannot be thought of as being continuous. The second is that we are interested in multivalued maps because we use them as outer representations of continuous maps. But it is obvious that there is no continuous map $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

To overcome these problems we need to introduce a notion of continuity for multivalued maps. Recall that for single valued functions, continuity is defined in terms of the pre-images of open sets. We want to do something similar for multivalued maps. However, the first problem is that there are at least two reasonable ways to define a pre-image.

Let $F : X \rightrightarrows Y$ and let $A \subset X$, $B \subset Y$. We define the *image of* A by

$$F(A) := \bigcup_{x \in A} F(x).$$

The *weak pre-image* of B under F is

$$F^{*-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\},$$

while the *pre-image* of B is

$$F^{-1}(B) := \{x \in X \mid F(x) \subset B\}.$$

Definition 6.16 A multivalued map F is *upper semicontinuous* if $F^{-1}(U)$ is open for any open $U \subset Y$ and it is *lower semicontinuous* if the set $F^{*-1}(U)$ is open for any open $U \subset Y$.

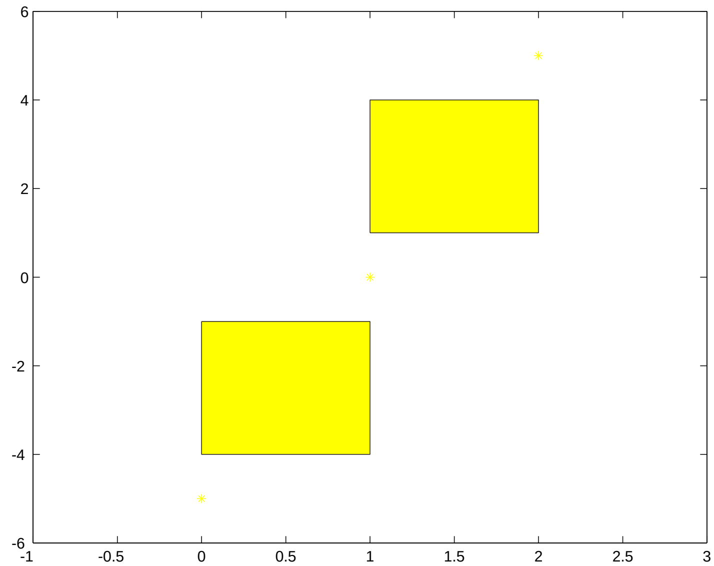


Fig. 6.1. The graph of the cubical map F.

Proposition 6.17 *Assume $F : X \rightrightarrows Y$ is a cubical map. F is lower semicontinuous if and only if the following property is satisfied:*

$$\text{If } P, Q \in \mathcal{K}(X) \text{ are such that } P \prec Q, \text{ then } F(\overset{\circ}{P}) \subset F(\overset{\circ}{Q}). \tag{6.8}$$

Proof. Suppose that F is lower semicontinuous. Since $F(\overset{\circ}{Q}) = F(x)$ for $x \in \overset{\circ}{Q}$, the set $F(\overset{\circ}{Q})$ is cubical and consequently closed. Thus the set $U := Y \setminus F(\overset{\circ}{Q})$ is open. By the lower semicontinuity of F,

$$V := F^{*-1}(U) = \{z \in X \mid F(z) \cap U \neq \emptyset\} \tag{6.9}$$

is open.

Now consider $x \in \overset{\circ}{P}$. Since F is cubical, $F(x) = F(\overset{\circ}{P})$. Therefore, it is sufficient to prove that $F(x) \subset F(\overset{\circ}{Q})$. This is equivalent to showing that $x \notin V$, which will be proved by contradiction.

So, assume that $x \in V$. Since $x \in \overset{\circ}{P}$ and $P \prec Q$, it follows that $x \in Q = \operatorname{cl}(\overset{\circ}{Q})$. Thus, $V \cap \operatorname{cl}(\overset{\circ}{Q}) \neq \emptyset$. But V is open, hence $V \cap \overset{\circ}{Q} \neq \emptyset$. Let $z \in V \cap \overset{\circ}{Q}$. Then, because F is cubical, $F(z) = F(\overset{\circ}{Q})$, and hence, $F(z) \cap U = \emptyset$. Thus, $z \notin V$, a contradiction, which proves (6.8).

Suppose now that F has property (6.8). Let U be open in Y and let V be defined as in (6.9). We need to show that V is open. If $V = \emptyset$ this is true by definition of topology, so let $V \neq \emptyset$. It is enough to show that the open hull $\operatorname{oh}(x)$ is contained in V for any point x of V. Indeed, $\operatorname{oh}(x)$ is the union of elementary cells $\overset{\circ}{Q}$ such that $x \in Q$. Fix such a Q. Let P be the unique elementary cube such that $x \in \overset{\circ}{P}$. Then $P \subset Q$. Since $F(x) = F(\overset{\circ}{P}) \subset F(\overset{\circ}{Q})$, it follows that $F(\overset{\circ}{Q}) \cap U \neq \emptyset$. Since F is constant on open cubes, it follows that $F(z) \cap U \neq \emptyset$ for all $z \in \overset{\circ}{Q}$. Thus $\overset{\circ}{Q} \subset V$. Since this is true for all $\overset{\circ}{Q} \subset \operatorname{oh}(x)$, the conclusions follows. □

Here is a dual property of upper semicontinuous maps.

Proposition 6.18 *Assume $F : X \rightrightarrows Y$ is a cubical map. Then F is upper semicontinuous if and only if it has the following property:*
For any $P, Q \in \mathcal{K}(X)$ such that P is a face of Q, $F(\overset{\circ}{Q}) \subset F(\overset{\circ}{P})$.

Proof. Exercise 6.6. □

Corollary 6.19 *The map $\lfloor \mathcal{F} \rfloor$ is lower semicontinuous and the map $\lceil \mathcal{F} \rceil$ is upper semicontinuous.*

The above propositions show how to construct lower semicontinuous and upper semicontinuous maps if their values on elementary cells are given. Indeed, if $F(\overset{\circ}{Q})$ is defined for all maximal Q and we put

$$F(\overset{\circ}{P}) = \bigcap \{F(\overset{\circ}{Q}) \mid P \prec Q \text{ and } Q \text{ is maximal}\},$$

we get a lower semicontinuous map. Similarly, if we put

$$F(\overset{\circ}{P}) = \bigcup \{F(\overset{\circ}{Q}) \mid P \prec Q \text{ and } Q \text{ is maximal}\},$$

we get an upper semicontinuous map.

Lower semicontiuous maps are those which we need for computing the homology, that is why we left the above Proposition as exercise. Upper semicontinous maps, however, are those whose graphs are easier to draw since we have the following:

Proposition 6.20 *Assume $F : X \rightrightarrows Y$ is a cubical map. Then F is upper semicontinuous if and only if its graph given by*

$$\mathrm{graph(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}}$$

is closed.

Proof. Exercise 6.7. □

Exercises

6.5 Let $X = [-1,1]^2 \subset \mathbf{R}^2$. Let $Y = [-2,2]^2 \subset \mathbf{R}^2$. Consider the map $A : X \to Y$ given by

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}.$$

Find a lower semicontinuous multivalued map $F : X \rightrightarrows Y$ with the property that $Ax \in F(x)$ for every $x \in X$. Try to find such a map with the smallest possible values.

6.6 Prove Proposition 6.18.

6.7 * Prove Proposition 6.20.

6.8 Let $F : [a,b] \rightrightarrows [c,d]$ be a cubical lower semicontinuous map (in particular, with acyclic values), where $a < b$ and $c < d$ are integers in $\mathbf{R}$. Prove that there exists a continuous single-valued map $f : [a,b] \to [c,d]$ such that $f(x) \in F(x)$ for all x.

6.3 Chain Selectors

As has been indicated since Chapter 5, our purpose for introducing multivalued maps is to obtain an outer representation for continuous functions. Of course, we still need to indicate how this outer representation can be used to generate homology. By Section 4.1 it is sufficient to indicate how a multivalued map can induce a chain map. In general the problem is difficult and even may not have a satisfactory solution. But there is a special class of multivalued maps, for which the answer to this question is relatively simple.

Definition 6.21 A cubical multivalued map $F : X \rightrightarrows Y$ is called *acyclic-valued* if for every $x \in X$ the set $F(x)$ is acyclic.

Theorem 6.22 *Assume $F : X \rightrightarrows Y$ is a lower semicontinuous acyclic-valued cubical map. Then, there exists a chain map $\varphi : C(X) \to C(Y)$ with the two properties*

$$|\varphi(\widehat{Q})| \subset F(\overset{\circ}{Q}) \text{ for all } Q \in \mathcal{K}(X), \tag{6.10}$$

$$\varphi(\widehat{Q}) \in \widehat{\mathcal{K}}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X), \tag{6.11}$$

Proof. We will construct the homomorphisms $\varphi_k : C_k(X) \to C_k(Y)$ by induction in k.

For $k < 0$, $C_k(X) = 0$, therefore there is no choice but to define $\varphi_k := 0$.

Consider $k = 0$. For each $Q \in \mathcal{K}_0$, choose $P \in \mathcal{K}_0(F(Q))$ and set

$$\varphi_0(\widehat{Q}) := \widehat{P}. \tag{6.12}$$

Clearly, $|\varphi_0\widehat{Q}| = P \in F(Q)$. Since, $Q \in \mathcal{K}_0$, $\overset{\circ}{Q}= Q$ and hence $F(Q) = F(\overset{\circ}{Q})$. Therefore,

$$|\varphi_0\widehat{Q}| \subset F(\overset{\circ}{Q}).$$

Furthermore,

$$\varphi_{-1}\partial_0 = 0 = \partial_0\varphi_0.$$

To continue the induction, suppose now that the homomorphisms $\varphi_i : C_i(X) \to C_i(Y)$, $i = 0, 1, 2, \ldots, k-1$, are constructed in such a way that

$$|\varphi_i\widehat{Q}| \subset F(\overset{\circ}{Q}) \text{ for all } Q \in \mathcal{K}_i(K),$$

and

$$\varphi_{i-1}\partial_i = \partial_i\varphi_i. \tag{6.13}$$

Let $\widehat{Q} \in \mathcal{K}_k(X)$. Then $\partial\widehat{Q} = \sum_{j=1}^{m} \alpha_j\widehat{Q}_j$ for some $\alpha_j \in \mathbf{Z}$ and $\widehat{Q}_j \in \mathcal{K}_{k-1}(X)$. Since F is lower semicontinuous, we have by Proposition 6.17

$$|\varphi_{k-1}\widehat{Q}_j| \subset F(\overset{\circ}{Q}_j) \subset F(\overset{\circ}{Q})$$

for all $j = 1, \ldots, m$. Thus

$$|\varphi_{k-1}\partial\widehat{Q}| \subset F(\overset{\circ}{Q}).$$

Since $F(\overset{\circ}{Q}) = F(x)$ for any $x \in\overset{\circ}{Q}$, the set $F(\overset{\circ}{Q})$ is acyclic. By the induction assumption (6.13)

$$\partial_{k-1}\varphi_{k-1}\partial_k\widehat{Q} = \varphi_{k-2}\partial_{k-1}\partial_k\widehat{Q} = 0,$$

i.e. $\varphi_{k-1}\partial\widehat{Q}$ is a cycle.

We should distinguish here the case $k = 1$. In that case, Q is an interval and

$$\partial\widehat{Q} = \widehat{B} - \widehat{A}$$

where A and B are vertices of Q. We showed above that the vertices $\varphi_0(A)$ and $\varphi_0(B)$ are supported in $F(\overset{\circ}{Q})$. Since $F(\overset{\circ}{Q})$ is acyclic, its reduced homology $\tilde{H}_*(F(\overset{\circ}{Q}))$ is trivial so, in particular, $\ker\epsilon = \operatorname{im}\partial_1$ in $\tilde{\mathcal{C}}(F(\overset{\circ}{Q}))$. Thus, there exists $c \in C_1(\varphi(\overset{\circ}{Q}))$ such that $\partial_1 c = \varphi_0(B) - \varphi_0(A) \in \ker\epsilon$. We put

$$\varphi_1\widehat{Q} := c.$$

When $k > 1$, the acyclicity of $F(\overset{\circ}{Q})$ implies that there exists a chain $c \in C_k(F(\overset{\circ}{Q}))$ such that $\partial c = \varphi_{k-1}\partial\widehat{Q}$. Define

$$\varphi_k \widehat{Q} := c.$$

By definition, the homomorphism φ_k satisfies the property

$$\partial_k \varphi_k = \varphi_{k-1} \partial_k.$$

Also, if $Q \in \mathcal{K}_k(X)$, then $\varphi_k \widehat{Q} \in C_k(F(\overset{\circ}{Q}))$, hence

$$|\varphi_k \widehat{Q}| \subset F(\overset{\circ}{Q}).$$

Therefore the chain map $\varphi = \{\varphi_k\}_{k \in \mathbf{Z}} : C(X) \to C(Y)$ satisfying (6.10) is well defined. □

Observe that in the first nontrivial step (6.12) of the inductive construction of φ we were allowed to choose any $P \in \mathcal{K}_0(F(Q))$. Thus, this procedure allows us to produce many chain maps of the type described in Theorem 6.22. This leads to the following definition.

Definition 6.23 A chain map $\varphi : C(X) \to C(Y)$ satisfying the conditions (6.10) and (6.11) in Theorem 6.22:

$$|\varphi(\widehat{Q})| \subset F(\overset{\circ}{Q}) \text{ for all } Q \in \mathcal{K}(X),$$

$$\varphi(\widehat{Q}) \in \widehat{\mathcal{K}}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X),$$

is called a *chain selector* of F.

Proposition 6.24 *Assume $F : X \rightrightarrows Y$ is a lower semicontinuous cubical map and φ is a chain selector for F. Then, for any $c \in C(X)$*

$$|\varphi(c)| \subset F(|c|).$$

Proof. Let $c = \sum_{i=1}^m \alpha_i \widehat{Q}_i$, where $\alpha_i \in \mathbf{Z}$, $\alpha_i \neq 0$. Then

$$\begin{aligned} |\varphi(c)| &= |\sum_{i=1}^m \alpha_i \varphi(\widehat{Q}_i)| \\ &\subset \bigcup_{i=1}^m |\varphi(\widehat{Q}_i)| \\ &\subset \bigcup_{i=1}^m F(\overset{\circ}{Q}_i) \\ &\subset \bigcup_{i=1}^m F(Q_i) \\ &= F(\bigcup_{i=1}^m Q_i) = F(|c|). \end{aligned}$$

□

Theorem 6.22 indicates that for appropriate multivalued maps we can produce a chain selector, but as was observed above this chain selector need not be unique. Of course, our primary interest is not in the chain selector, but rather in the homology maps that they induce. The following theorem, shows that every chain selector leads to the same map on homology.

Theorem 6.25 *Let $\varphi, \psi : C(X) \to C(Y)$ be chain selectors for the lower semicontinous acyclic-valued cubical map $F : X \rightrightarrows Y$. Then φ is chain homotopic to ψ and hence they induce the same homomorphism in homology.*

Proof. A chain homotopy $D = \{D_k : C_k(X) \to C_{k+1}(Y)\}_{k\in\mathbf{Z}}$ joining φ to ψ can be constructed by induction.

For $k < 0$, there is no choice but to set $D_k := 0$. Thus assume $k \geq 0$ and D_i is defined for $i < k$ in such a way that

$$\partial_{i+1} \circ D_i + D_{i-1} \circ \partial_i = \psi_i - \varphi_i, \tag{6.14}$$

and for all $Q \in \mathcal{K}_i(X)$ and $c \in C_i(Q)$,

$$|D_i(c)| \subset F(\overset{\circ}{Q}). \tag{6.15}$$

Let $\widehat{Q} \in C_k(X)$ be an elementary k-cube. Put

$$c := \psi_k(\widehat{Q}) - \varphi_k(\widehat{Q}) - D_{k-1}\partial_k(\widehat{Q}).$$

It follows easily from the induction assumption that c is a cycle. Moreover, if $\partial\widehat{Q} = \sum_{i=1}^{m} \alpha_i \widehat{P}_i$ for some $\alpha_i \neq 0$ and $P_i \in \mathcal{K}_{k-1}(X)$, then P_i are faces of Q and by Proposition 6.17

$$|D_{k-1}\partial(\widehat{Q})| \subset \bigcup_{i=1}^{m} |D_{k-1}(\widehat{P}_i)| \subset \bigcup_{i=1}^{m} F(\overset{\circ}{P}_i) \subset F(\overset{\circ}{Q}).$$

Consequently

$$|c| \subset |\psi_k(\widehat{Q})| \cup |\varphi_k(\widehat{Q})| \cup |D_{k-1}\partial_k(\widehat{Q})| \subset F(\overset{\circ}{Q}).$$

It follows that $c \in Z_k(F(\overset{\circ}{Q}))$.

Now, since $F(\overset{\circ}{Q})$ is acyclic, we conclude for $k > 0$ that there exists a $c' \in C_{k+1}(F(\overset{\circ}{Q}))$ such that $\partial c' = c$. In the case $k = 0$, the same conclusion follows as in the proof of Theorem 6.22 from the identity $\tilde{H}_*(F(\overset{\circ}{Q})) = 0$ and from the equation

$$c := \psi_0(\widehat{Q}) - \varphi_0(\widehat{Q}) - D_{-1}\partial_0(\widehat{Q}) = \psi_0(\widehat{Q}) - \varphi_0(\widehat{Q}).$$

We put $D_k(\widehat{Q}) := c'$. One easily verifies that the induction assumptions are satisfied, therefore the construction of the required homotopy is completed. □

The above theorem lets us make the following fundamental definition.

Definition 6.26 Let $F : X \rightrightarrows Y$ be a lower semicontinuous acyclic-valued cubical map. Let $\varphi : C(X) \to C(Y)$ be a chain selector of F. The *homology map* of F, $F_* : H_*(X) \to H_*(Y)$ is defined by

$$F_* := \varphi_*.$$

Example 6.27 Let X and Y be connected cubical sets, $F : X \rightrightarrows Y$ a lower semicontinuous acyclic-valued cubical map and $\varphi : C(X) \to C(Y)$ a chain selector of F. By Theorem 2.59, all vertices in a connected set are homologous so $H_0(X) = \mathbf{Z}[\widehat{V}]$ and $H_0(Y) = \mathbf{Z}[\widehat{W}]$ for arbitrarily chosen $V \in \mathcal{K}_0(X)$ and $W \in \mathcal{K}_0(Y)$. Since $\varphi_0(\widehat{V}) \in \widehat{\mathcal{K}}_0(Y)$ and $\varphi_0(\widehat{V}) \sim \widehat{W}$, we get

$$F_{*0}([\widehat{V}]) = \varphi_{*0}([\widehat{V}]) = [\varphi_0(\widehat{V})] = [\widehat{W}].$$

In particular, if $X = Y$ is connected then

$$F_{*0} = \mathrm{id}_{H_0(X)}.$$

Keep in mind that one of the purposes of introducing multivalued maps is to be able to compute the homology of a continuous map by some systematic method of representation. Obviously, and we saw this in Chapter 5, what procedure one uses or the amount of computation one is willing to do determines how sharp a representation one obtains. An obvious question is how much does this matter? In other words, given an acyclic-valued cubical map, does an acyclic-valued map with smaller values produce a different map on homology? To answer this we need to make the question precise.

Definition 6.28 Let X and Y be cubical spaces and let $F, G : X \rightrightarrows Y$ be lower semicontinuous cubical maps. F is a *submap* of G if

$$F(x) \subset G(x)$$

for every $x \in X$. This is denoted by $F \subset G$.

Proposition 6.29 *If $F, G : K \rightrightarrows L$ are two lower semicontinuous acyclic-valued cubical maps and F is a submap of G, then $F_* = G_*$.*

Proof. Let φ be a chain selector of F. Then, φ is also a chain selector of G. Hence, by definition

$$F_* = \varphi_* = G_*.$$

□

A fundamental property of maps is that they can be composed. In the case of multivalued maps $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ we will construct the multivalued map $G \circ F : X \rightrightarrows Z$, by setting

$$G \circ F(x) := G(F(x))$$

for every $x \in X$.

Proposition 6.30 *If $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ and $H : X \rightrightarrows Z$ are lower semicontinuous acyclic-valued cubical maps and $G \circ F \subset H$ then $H_* = G_* \circ F_*$.*

Proof. Let $\varphi \in C(F)$ and $\psi \in C(G)$. Then by Proposition 6.24 for any $Q \in \mathcal{K}(X)$

$$|(\psi(\varphi(\widehat{Q}))| \subset G(|\varphi(\widehat{Q})|) \subset G(F(\mathring{Q})) \subset H(\mathring{Q}).$$

Hence $\psi \circ \varphi \in C(H)$. But we can compose chain maps and hence

$$H_* = (\psi\varphi)_* = \psi_*\varphi_* = G_*F_*.$$

□

Corollary 6.31 *If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are lower semicontinuous acyclic-valued cubical maps and $G \circ F$ is acyclic-valued then $(G \circ F)_* = G_* \circ F_*$.*

6.4 Homology of Continuous Maps

We are finally in the position to discuss the homology of continuous maps. Recall the discussion of maps in Chapter 5. There we considered continuous functions of one variable and used Taylor's theorem to get bounds on images of elementary cubes by the function. While this provided a conceptually easy approach to getting bounds, Taylor polynomials are a rather inefficient means of approximating functions over a given region. Furthermore, for different problems, different methods will be the most appropriate. Therefore, we would like our discussion of the construction of the multivalued map to be independent of the particular method of approximation employed. This leads us to make the following definition.

Definition 6.32 Let X and Y be cubical sets and let $f : X \to Y$ be a continuous function. A *cubical representation* of f or simply a *representation* of f is a lower semicontinuous multivalued cubical map $F : X \rightrightarrows Y$ such that

$$f(x) \in F(x) \tag{6.16}$$

for every $x \in X$.

Assume that f has a cubical representation $F : X \rightrightarrows Y$ which is acyclic-valued. Then, the natural candidate for the definition of the associated homology map, $f_* : H_*(X) \to H_*(Y)$, is

$$f_* := F_*. \tag{6.17}$$

There are at least two questions that need to be answered before we can be content with this approach to defining the homology of a continuous map. First, observe that given a continuous function, there may be many cubical representations. Thus, we will need to show that all cubical representations of a given function give rise to the same homomorphism on homology. This will be the content of Section 6.4.1. Second, given cubical sets and a continuous map between them it need not be the case that there exists a cubical representation, which is acyclic-valued. We will deal with this problem in Section 6.4.2.

6.4.1 Cubical Representations

From the point of view of computations one typically wants a cubical representation whose images are as small as possible.

Proposition 6.33 *Let X and Y be cubical sets and let $f : X \to Y$ be a continuous function. The map $M_f : X \rightrightarrows Y$ defined by*

$$M_f(x) := \operatorname{ch}(f(\operatorname{ch}(x))) \tag{6.18}$$

is a cubical representation of f.

Proof. The fact that M_f is a cubical map follows from Proposition 6.10, properties (iii) and (ix). Since obviously $f(x) \in M_f(x)$, M_f is a representation of f. □

Definition 6.34 The map M_f is called the *minimal representation* of f . If M_f is acyclic-valued, then M_f is referred to as the *cubical carrier* of f.

Example 6.35 Consider the continuous function $f : [0,3] \to [0,3]$ given by $f(x) = x^2/3$. Figure 6.2 indicates the graph of f and its minimal cubical representation M_f. To verify that M_f really is the minimal cubical representation just involves checking the definition on all the elementary cubes in $[0,3]$. To begin with consider $[0] \in \mathcal{K}_0$. $\operatorname{ch}[0] = [0]$ and $f(0) = 0$, therefore $M_f(0) = 0$. On the other hand, while $\operatorname{ch}[1] = [1]$, $f(1) = 1/3 \in [0,1]$ and hence $\operatorname{ch} f(1) = [0,1]$. Therefore, $M_f([1]) = [0,1]$. The rest of the elementary cubes can be checked in a similar manner.

Observe that if any cube from the graph of M_f were removed, then the graph of f would no longer be contained in the graph of M_f. In this sense M_f is minimal.

More precisely we have the following proposition.

Proposition 6.36 *Let X and Y be cubical sets, let $f : X \to Y$ be a continuous function and let $F : X \rightrightarrows Y$ be a cubical representation to f. Then, M_f is a submap of F.*

Proof. Let $x \in X$ and let $Q \in \mathcal{K}(X)$ be such that $x \in \mathring{Q}$. Since F is a cubical map, $F(x) = F(\mathring{Q})$ and in particular, $F(x)$ is closed. Let $\bar{x} \in Q$. By Proposition 2.15(iv) there exists a sequence $\{x_n\} \subset \mathring{Q}$ such that $x_n \to \bar{x}$. By continuity of f, $f(\bar{x}) \in F(\mathring{Q})$ which implies that $f(Q) \subset F(\mathring{Q})$ and also $\operatorname{ch} f(Q) \subset F(\mathring{Q})$, because $F(\mathring{Q})$ is a cubical set. Since $x \in \mathring{Q}$, $\operatorname{ch}(x) = Q$. Thus,

$$M_f(x) = \operatorname{ch}(f(\operatorname{ch}(x))) = \operatorname{ch}(f(Q)) \subset F(\mathring{Q}) = F(x).$$

□

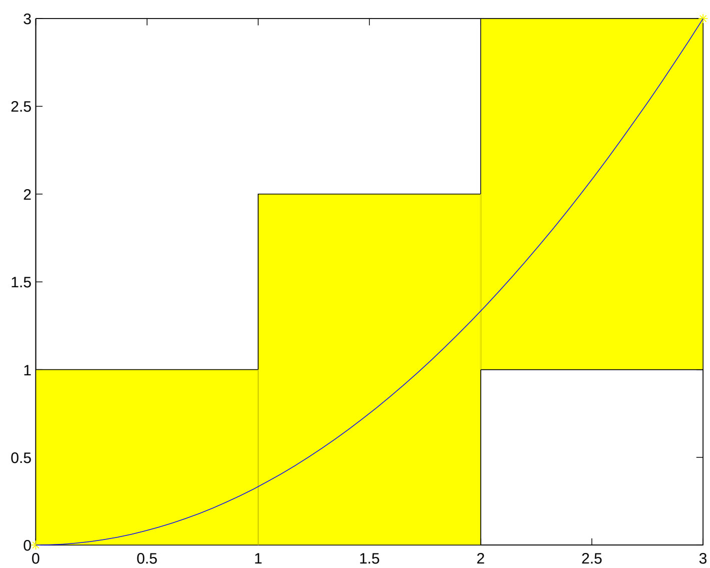


Fig. 6.2. The graph of the continuous function $f : [0, 3] \to [0, 3]$ and the graph of M_f.

Example 6.37 As is suggested by the previous definition, it is not true that a minimal representation is necessarily acyclic-valued. Let $X := \Gamma^1$, where Γ^1 is the boundary of the unit circle considered in Section 5.4. Note that X is the union of the elementary cubes:

$$K_1 := [0] \times [0,1] \quad K_2 := [0,1] \times [1]$$
$$K_3 := [1] \times [0,1] \quad K_4 := [0,1] \times [0].$$

Define the map $\lambda : [0,1] \to X$ for $s \in [0,1]$ by

$$\lambda(t) := \begin{cases} (0, 4s) & \text{if } s \in [0, 1/4] \\ (4s - 1, 1) & \text{if } s \in [1/4, 1/2] \\ (1, 3 - 4s) & \text{if } s \in [1/2, 3/4] \\ (4 - 4s, 0) & \text{if } s \in [3/4, 1] \end{cases}$$

and the map $f : X \to X$ for $(x_1, x_2) \in X$ by

$$f(x_1, x_2) := \begin{cases} \lambda(x_2) & \text{if } (x_1, x_2) \in K_1 \cup K_3 \\ \lambda(x_1) & \text{if } (x_1, x_2) \in K_2 \cup K_4. \end{cases}$$

Since the local formulas for $\lambda(t)$ and $f(x_1, x_2)$ agree at the points where the consecutive intervals intersect, f is continuous. For any $(x_1, x_2) \in \overset{\circ}{K}_i$

$$M_f(x_1, x_2) = \operatorname{ch}(f(\operatorname{ch}(x_1, x_2))) = \operatorname{ch}(f(K_i)) = \operatorname{ch}(X) = X.$$

Since X is not acyclic, it follows that M_f is not acyclic-valued.

In order to visualize better the function f, we rephrase the formula in the language used in Section 5.4.

Recall from Section 5.4 that any point $x \in \Gamma^1$ can be identified with $t \in [0,4]$ by the function

$$t \mapsto x(t) = (x_1(t), x_2(t)),$$

defined in the formula (5.5), upon the identification $t \sim t + 4$. Also recall that $t = 0, 1, 2, 3$ and $4 \sim 0$ correspond to elementary vertices of Γ^1 while the intervals $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$ correspond to, respectively, K_4, K_3, K_2, K_1, by going counterclockwise around the square $[0,1]^2$.

Now our map $f : X \to X$ can be given an alternative presentation in terms of the parameter $t \in [0,4]$:

$$t \mapsto \begin{cases} 4t & \text{if } t \in [0,2], \\ 16 - 4t & \text{if } t \in [2,4]. \end{cases}$$

which looks like a tent map of slope 4. Of course the values in the above formula fall outside of the interval $[0,4]$, but the identification $t \sim t+4$ breaks the formula to four cases:

$$f(x(t)) := \begin{cases} x(4t) & \text{if } t \in [0,1], \\ x(4t - 4) & \text{if } t \in [1,2], \\ x(12 - 4t) & \text{if } t \in [2,3], \\ x(16 - 4t) & \text{if } t \in [3,4]. \end{cases}$$

The graph of f as a function of t is illustrated in Figure 6.3.

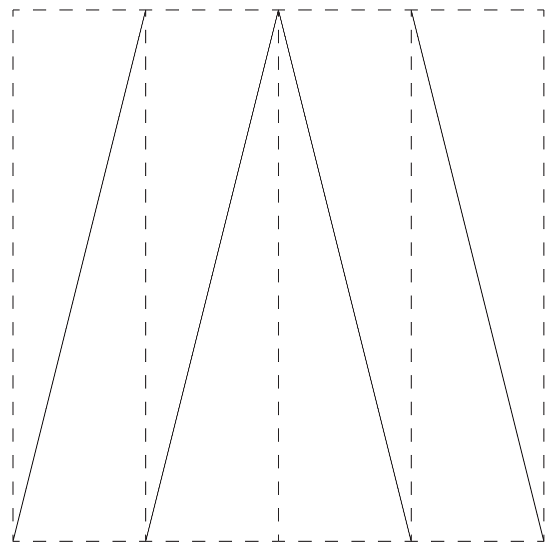


Fig. 6.3. The graph of f in Example 6.37 presented as a function of the parameter $t \in [0, 4]$. At the first sight it doesn't look like a graph of a continuous function but keep in mind that the endpoints of vertical dotted lines are identified by the relation $t \sim t + 4$.

Proposition 6.38 *Let X and Y be cubical sets, let $f : X \to Y$ be a continuous function such that its minimal representation is acyclic-valued. If $F, G : X \rightrightarrows Y$ are any other acyclic-valued representations of f then $F_* = G_*$.*

Proof. By Proposition 6.36, M_f is a submap of F and a submap of G. Hence it follows from Proposition 6.29 that

$$F_* = (M_f)_* = G_*.$$

□

The last proposition implies that if the minimal representation of f is acyclic-valued, then (6.17) is well defined. Now that we have a formula to work with we can, using some very simple examples, try to reinforce the intuition about homology maps that we began to develop in Chapter 5. We begin with a technical result.

Proposition 6.39 *Let $X \subset \mathbf{R}^d$ be a cubical set. Let $A \subset X$ be such that* $\operatorname{diam} A < 1$*. Then,* $\operatorname{ch}(A) \cap X$ *is acyclic.*

Proof. Let

$$\mathcal{Q} := \{Q \in \mathcal{K}(X) \mid \overset{\circ}{Q} \cap A \neq \emptyset\}.$$

Since X is cubical

$$\operatorname{ch}(A) \cap X = \bigcup_{Q \in \mathcal{Q}} Q.$$

Observe that for any two elementary cubes $P, Q \in \mathcal{Q}$ the intersection $P \cap Q$ is non-empty, because otherwise $\operatorname{diam} A \geq 1$. Therefore by Proposition 2.85, $\bigcap \mathcal{Q}$ is non-empty. It follows that $\operatorname{ch}(A)$ is star-shaped and consequently acyclic by Proposition 2.84. □

The simplest non-trivial map is the identity.

Proposition 6.40 *Let X be a cubical set. Consider the identity map $\operatorname{id}_X : X \to X$. Then, M_{id_X} is acyclic-valued and*

$$(\operatorname{id}_X)_* = \operatorname{id}_{H_*(X)}$$

Proof. By Proposition 6.10

$$M_{\operatorname{id}_X}(x) = \operatorname{ch}(x),$$

which, by Proposition 6.39 is acyclic. Therefore, M_{id_X} is a cubical carrier of id_X and

$$(\operatorname{id}_X)_* = \left(M_{\operatorname{id}_X}\right)_*.$$

Let $Q \in \mathcal{K}(X)$. Then

$$|\operatorname{id}_{C(X)}(\widehat{Q})| = Q = \operatorname{ch}(\overset{\circ}{Q}) = M_{\operatorname{id}_X}(\overset{\circ}{Q}).$$

Therefore, $\operatorname{id}_{C(X)}$ is a chain selector for M_{id_X}. Finally, it is easy to check that $\operatorname{id}_{C(X)}$ induces the identity map $\operatorname{id}_{H_*(X)}$ on homology. □

Lemma 6.41 *Let X, Y, and Z be cubical sets. Assume $f : X \to Y$ and $g : Y \to Z$ are continuous maps such that M_f, M_g, $M_{g \circ f}$ and $M_g \circ M_f$ are acyclic-valued. Then,*

$$(g \circ f)_* = g_* \circ f_*$$

Proof. Observe that

$$M_{g \circ f}(x) = \operatorname{ch}(g(f(\operatorname{ch}(x)))) \subset \operatorname{ch}(g(\operatorname{ch}(f(\operatorname{ch}(x))))) = M_g(M_f(x)),$$

i.e. $M_{g \circ f} \subset M_g \circ M_f$. Therefore, from Propositions 6.29 and Corollary 6.31

$$(g \circ f)_* = (M_{g \circ f})_* = (M_g \circ M_f)_* = (M_g)_* \circ (M_f)_* = g_* \circ f_*.$$

□

The last lemma is a temporary tool. In the next section we will present a general result, Theorem 6.58, where we will get rid of the technical assumptions about the acyclicity of the values of M_f, M_g, $M_{g\circ f}$ and $M_g \circ M_f$. However, before we prove Theorem 6.58, we have to make sure that these four assumptions are satisfied whenever we want to compose homology of maps.

Example 6.37 showed us that M_g does not need to by acyclic-valued, therefore we are not done yet with the definition of homology of a map. There are, however, some simple but useful maps whose minimal representations are acyclic-valued and so we already know their homology. These maps were studied in Chapter and we are in position now to conclude that the related chain maps presented there are in fact chain selectors of minimal representations. We start from the following example.

Example 6.42 $A \subset X$ are cubical sets in $\mathbf{R}^d$. In Example 4.4 we pointed out that the inclusion map $i : A \hookrightarrow X$ gives rise to an inclusion map of the chain complexes, $\iota : \mathcal{C}(A) \hookrightarrow \mathcal{C}(X)$, defined by

$$\iota_k \widehat{Q} = \widehat{Q} \quad \text{for all } Q \in \mathcal{K}_k(A).$$

We are now able to precise what does did we mean by "gives rise to". Since $i(Q) = Q$ for every elementary cube $Q \in A$, M_i is defined by

$$M_i(x) = \operatorname{ch}(ch(x)) = ch(x)$$

for all $x \in A$. From this,

$$M_i(\overset{\circ}{Q}) = Q.$$

for all $Q \in \mathcal{K}(A)$. Hence the chain inclusion ι satisfies the two conditions defining a chain selector of M_i and $i_* := (M_i)_* = \iota_*$.

Example 6.43 Let f be any of the following maps:

(a) The immersion $j : X \to \{0\} \times X$ discussed in Example 4.6,
(b) The translation $t : X \to \mathbf{m} + X$ discussed in Example 4.7,
(b) The projection $p : Q \to Q$ discussed in Example 4.8.

We leave as an exercise checking that the minimal representation of f has the property

$$M_f(x) = f(\operatorname{ch}(x))$$

and verifying that the previously defined related chain map, respectively κ, τ, and π, is indeed a chain selector of M_f.

6.4.2 Rescaling

So far we are able to define the homology map of a continuous function when its minimal representation is acyclic-valued. Unfortunately, as was indicated in Example 6.37 this is not true for every continuous function. We encountered this problem before in Section 5.4. In Section 5.2 we considered the procedure of subdividing the intervals. We could do the same thing here, i.e. we could try to make the images of all elementary cubes acyclic by subdividing the domain of the map into smaller cubes. However, that would require developing the homology theory for cubical sets defined on fractional grids. Obviously, this could be done, but it is not necessary. Instead we take an equivalent path based on rescaling the domain of the function to a large size already considered in Section 5.2. Observe that if we make the domain large, then as a fraction of the size of the domain the elementary cubes become small. This leads to the following notation.

Definition 6.44 A *scaling vector* is a vector of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbf{Z}^d$$

and gives rise to the *scaling* $\Lambda^\alpha : \mathbf{R}^d \to \mathbf{R}^d$ defined by

$$\Lambda^\alpha(x) := (\alpha_1 x_1, \alpha_2 x_2, \ldots \alpha_d x_d).$$

If $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ is another scaling vector, then set

$$\alpha\beta := (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_d\beta_d).$$

The following properties of scalings are straightforward and left as an exercise.

Proposition 6.45 *Let α and β be a scaling vector. Then, Λ^α maps cubical sets onto cubical sets and $\Lambda^\beta \circ \Lambda^\alpha = \Lambda^{\alpha\beta}$.*

Definition 6.46 Let $X \subset \mathbf{R}^d$ be a cubical set and let $\alpha \in \mathbf{Z}^d$ be a scaling vector. Define $\Lambda^\alpha_X := \Lambda^\alpha|_X$. The *scaling of* X by α is

$$X^\alpha := \Lambda^\alpha_X(X) = \Lambda^\alpha(X).$$

Example 6.47 Recall that Example 6.37 described a function f for which M_f was not a cubical representation. The first step in dealing with this problem involves rescaling the space X. Figure 6.4 shows the effect of scaling X using the scaling vector $\alpha = (4, 4)$.

We begin by establishing that scalings are nice continuous maps in the sense that their minimal cubical representations are acyclic-valued.

Proposition 6.48 *Let X be a cubical set and let α be a scaling vector. The map $M_{\Lambda^\alpha_X}$ is acyclic-valued.*

Proof. By definition, for any $x \in X$

$$M_{\Lambda^\alpha_X}(x) = (\operatorname{ch}(\Lambda^\alpha_X(\operatorname{ch}(x)))).$$

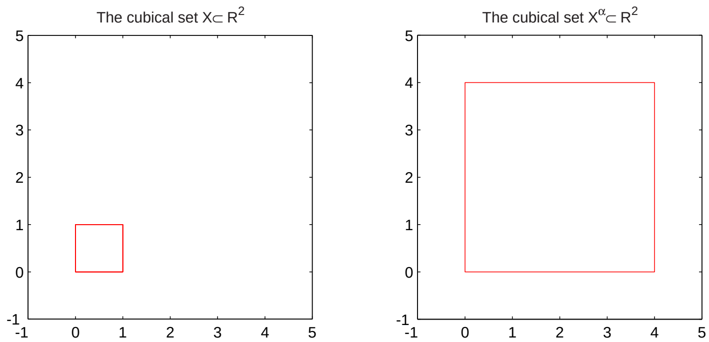


Fig. 6.4. The space X and X^α where $\alpha = (4,4)$.

Since $\operatorname{ch}(x)$ is a cube, it follows that $\Lambda^\alpha_X(\operatorname{ch}(x))$ is a cubical rectangle and consequently $\operatorname{ch}(\Lambda^\alpha_X(\operatorname{ch}(x))) = \Lambda^\alpha_X(\operatorname{ch}(x))$ is also a cubical rectangle. Therefore, by Proposition 2.81 the set $M_{\Lambda^\alpha_X}(x)$ is acyclic. □

In the sequel we will have to compose homology of scalings. In order to be able to apply Lemma 6.41 we need the following proposition.

Proposition 6.49 *Let X, Y, and Z be cubical sets and let α and β be scaling vectors. If $\Lambda^\alpha(X) \subset Y$ and $\Lambda^\beta(Y) \subset Z$, then $M_{\Lambda^\beta_Y \circ \Lambda^\alpha_X}$ and $M_{\Lambda^\beta_Y} \circ M_{\Lambda^\alpha_X}$ are acyclic-valued.*

Proof. Since $\Lambda^\beta_Y \circ \Lambda^\alpha_X = \Lambda^{\alpha\beta}_X$, the map $M_{\Lambda^\beta_Y \circ \Lambda^\alpha_X}$ is acyclic-valued by Proposition 6.48. To show that $M_{\Lambda^\beta_Y} \circ M_{\Lambda^\alpha_X}$ is acyclic-valued, observe that

$$M_{\Lambda^\beta_Y} \circ M_{\Lambda^\alpha_X}(x) = (\operatorname{ch}(\Lambda^\beta_Y(\operatorname{ch}(\Lambda^\alpha_X(\operatorname{ch}(x)))))).$$

Therefore the acyclicity of the values of $M_{\Lambda^\beta_Y} \circ M_{\Lambda^\alpha_X}$ follows by the argument similar as in the proof of Proposition 6.48. □

Since the minimal representations of scalings are acyclic-valued, they induce maps on homology. Furthermore, since scalings just change the size of the space one would expect that they induce isomorphisms on homology. The simplest way to check this is to show that their homology maps have inverses. Therefore, given a cubical set X and a scaling vector α let $\Omega^\alpha_X : X^\alpha \to X$ be defined by

$$\Omega^\alpha_X(x) := (\alpha_1^{-1}x_1, \alpha_2^{-1}x_2, \ldots, \alpha_d^{-1}x_d).$$

Obviously, $\Omega^\alpha_X = (\Lambda^\alpha_X)^{-1}$. However, we need to know that it induces a map on homology.

Lemma 6.50 $M_{\Omega^\alpha_X} : X^\alpha \rightrightarrows X$ *is acyclic-valued.*

Proof. Let $x \in X^\alpha$. Then $\mathrm{ch}\,(x) = P$ for some $P \in \mathcal{K}_k(X^\alpha)$ and

$$M_{\Omega_X^\alpha}(x) = \mathrm{ch}\,(\Omega_X^\alpha(\mathrm{ch}\,(x))) = \mathrm{ch}\,(\Omega_X^\alpha(P)).$$

Since obviously $\Omega_X^\alpha(P)$ is a Cartesian product of bounded intervals, the conclusion follows from Corollary 6.13 and Proposition 2.81 □

Proposition 6.51 *If X is a cubical set and α is a scaling vector, then*

$$(\Lambda_X^\alpha)_* : H_*(X) \to H_*(X^\alpha) \quad \text{and} \quad (\Omega_X^\alpha)_* : H_*(X^\alpha) \to H_*(X)$$

are isomorphisms. Furthermore,

$$(\Lambda_X^\alpha)_*^{-1} = (\Omega_X^\alpha)_*.$$

Proof. It follows from Proposition 6.48 and Lemma 6.50 that $M_{\Lambda_X^\alpha}$, $M_{\Omega_X^\alpha}$, $M_{\Omega_X^\alpha \circ \Lambda_X^\alpha} = M_{\mathrm{id}\,_X}$ and $M_{\Lambda_X^\alpha \circ \Omega_X^\alpha} = M_{\mathrm{id}\,_{X^\alpha}}$ are acyclic-valued. Moreover, for any $x' \in X^\alpha$

$$M_{\Lambda_X^\alpha} \circ M_{\Omega_X^\alpha}(x) = \mathrm{ch}\,(\Lambda_X^\alpha(\mathrm{ch}\,(\Omega_X^\alpha(\mathrm{ch}\,(x'))))) = \Lambda_X^\alpha(\mathrm{ch}\,(\Omega_X^\alpha(\mathrm{ch}\,(x')))).$$

By Corollary 6.13, $\mathrm{ch}\,(\Omega_X^\alpha(\mathrm{ch}\,(x')))$ is a rectangle. Hence, $M_{\Lambda_X^\alpha} \circ M_{\Omega_X^\alpha}(x)$ is a rectangle, since it is the rescaling of a rectangle. It follows from Proposition 2.81 that it is acyclic. Similarly for any $x \in X$

$$M_{\Omega_X^\alpha} \circ M_{\Lambda_X^\alpha}(x) = \mathrm{ch}\,(\Omega_X^\alpha(\mathrm{ch}\,(\Lambda_X^\alpha(\mathrm{ch}\,(x))))) = \mathrm{ch}\,(\Omega_X^\alpha(\Lambda_X^\alpha(\mathrm{ch}\,(x)))) = \mathrm{ch}\,(x)$$

is acyclic. Hence, by Lemma 6.41, and 6.40,

$$(\Lambda_X^\alpha)_* \circ (\Omega_X^\alpha)_* = (\Lambda_X^\alpha \circ \Omega_X^\alpha)_* = \mathrm{id}\,_{X^\alpha *} = \mathrm{id}\,_{H_*(X^\alpha)}$$

and

$$(\Omega_X^\alpha)_* \circ (\Lambda_X^\alpha)_* = (\Omega_X^\alpha \circ \Lambda_X^\alpha)_* = \mathrm{id}\,_{X*} = \mathrm{id}\,_{H_*(X)}.$$

□

As was indicated in the introduction, the purpose of scaling is to allow us to define the homology of an arbitrary continuous map between cubical sets. Thus, given a continuous map $f : X \to Y$ and a scaling vector α define

$$f^\alpha := f \circ \Omega_X^\alpha$$

Observe that $f^\alpha : X^\alpha \to Y$.

Example 6.52 To indicate the relationship between f and f^α we return to Example 6.37. Consider $\alpha = (4,4)$. As was already mentioned Figure 6.4 shows X^α and $f^\alpha : X^\alpha \to X$. Now consider M_{f^α}. Consider $Q = [0,1] \times [4]$. Let $(x_1, x_2) \in \overset{\circ}{Q}$. Then

$$\begin{aligned} M_{f^\alpha}(x_1,x_2) &= \mathrm{ch}\,(f^\alpha(\mathrm{ch}\,(x_1,x_2)) \\ &= \mathrm{ch}\,(f^\alpha(Q)) \\ &= \mathrm{ch}\,(\lambda([0,1/4])) \\ &= \mathrm{ch}\,([0]\times[0,1]) \\ &= [0]\times[0,1] \end{aligned}$$

which is acyclic. Similar checks at all the points on X^α shows that M_{f^α} is acyclic-valued and hence M_{f^α} is a cubical representation.

Theorem 6.53 *Let X and Y be cubical sets and $f : X \to Y$ be continuous. Then there exists a scaling vector α such that M_{f^α} is acyclic-valued. Moreover, if β is another scaling vector such that M_{f^β} is acyclic-valued, then*

$$f^\alpha_*(\Lambda^\alpha_X)_* = f^\beta_*(\Lambda^\beta_X)_*$$

Proof. The continuity of f lets us choose $\delta > 0$ such that for $x, y \in X$

$$\mathrm{dist}\,(x,y) \le \delta \quad \Rightarrow \quad \mathrm{dist}\,(f(x), f(y)) \le \frac{1}{2} \tag{6.19}$$

Let α be a scaling vector such that $\min\{\alpha_i \mid i = 1,\ldots,n\} \ge 1/\delta$. Since $\mathrm{diam}\,\mathrm{ch}\,(x) \le 1$, we get from (6.19) that

$$\mathrm{diam}\, f^\alpha(\mathrm{ch}\,(x)) \le \frac{1}{2}.$$

Therefore it follows from Proposition 6.39 that M_{f^α} is acyclic-valued.

Now assume that the scaling vector β is such that M_{f^β} is also acyclic-valued. Also, assume for the moment that for each $i = 1,\ldots,n$, α divides β. Let $\gamma_i := \frac{\beta_i}{\alpha_i}$. Then $\gamma = (\gamma_1,\ldots,\gamma_d)$ is a scaling vector. Clearly, $\Lambda^\beta_X = \Lambda^\gamma_{X^\alpha} \circ \Lambda^\alpha_X$. Moreover, $M_{\Lambda^\gamma_{X^\alpha}} \circ M_{\Lambda^\alpha_X}$ is easily checked to be acyclic-valued. Therefore, it follows from Lemma 6.41 that

$$(\Lambda^\beta_X)_* = (\Lambda^\gamma_{X^\alpha})_* \circ (\Lambda^\alpha_X)_*.$$

On the other hand we also have

$$f^\alpha = f^\beta \circ \Lambda^\gamma_{X^\alpha},$$

hence

$$M_{f^\beta}(M_{\Lambda^\gamma_{X^\alpha}}(x)) = \mathrm{ch}\,(f^\beta(\mathrm{ch}\,(\Lambda^\gamma_{X^\alpha}(\mathrm{ch}\,(x))))) = \mathrm{ch}\,(f^\alpha(\mathrm{ch}\,(x))) = M_{f^\alpha}(x).$$

Thus, $M_{f^\beta} \circ M_{\Lambda^\gamma_{X^\alpha}}$ is acyclic-valued and we get from Lemma 6.41 that

$$f_*^\alpha = f_*^\beta \circ (\Lambda^\gamma_{X^\alpha})_*.$$

Consequently,

$$f_*^\alpha \circ (\Lambda^\alpha_X)_* = f_*^\beta \circ (\Lambda^\gamma_{X^\alpha})_* \circ (\Lambda^\alpha_X)_* = f_*^\beta \circ (\Lambda^\beta_X)_*.$$

Finally, if it is not true that α divides β for each $i = 1, \ldots, n$, then let $\theta = \alpha\beta$. By what we have just proven

$$f_*^\alpha \circ (\Lambda^\alpha_X)_* = f_*^\theta \circ (\Lambda^\theta_X)_* = f_*^\beta \circ (\Lambda^\beta_X)_*$$

which settles the general case. □

We can now give the general definition for the homology map of a continuous function.

Definition 6.54 Let X and Y be cubical sets and let $f : X \to Y$ be a continuous function. Let α be a scaling vector such that M_{f^α} is acyclic-valued. Then, $f_* : H_*(X) \to H_*(Y)$ is defined by

$$f_* := f_*^\alpha \circ (\Lambda^\alpha_X)_*.$$

The definition of f_* can be illustrated by diagrams as follows. First, f^α is defined so to complete the following commutative diagram.

$$\begin{array}{ccc} X^\alpha & \xrightarrow{f^\alpha} & Y \\ \downarrow{\scriptstyle \Omega^\alpha} & & \uparrow{\scriptstyle =} \\ X & \xrightarrow{f} & Y \end{array}$$

Next, f_* is defined so to complete the diagram below.

$$\begin{array}{ccc} H_*(X^\alpha) & \xrightarrow{f_*^\alpha = (M_{f^\alpha})_*} & H_*(Y) \\ \uparrow{\scriptstyle \Lambda_*^\alpha} & & \downarrow{\scriptstyle =} \\ H_*(X) & \xrightarrow{f_*} & H_*(Y) \end{array}$$

By Theorem 6.53, this definition is independent of the particular scaling vector used. However, we need to reconcile this definition of the homology map with that of equation (6.17). So assume that $f : X \to Y$ is such that M_f is acyclic-valued. Let α be the scaling vector where each $\alpha_i = 1$. Then $f^\alpha = f$, $\Lambda^\alpha_X = \mathrm{id}_X$ and

$$f_* = f_* \circ (\mathrm{id}_X)_* = (f^\alpha)_* \circ (\Lambda^\alpha_X)_*,$$

hence the two definitions of f_* agree.

Several more natural questions can be posed. We know from Proposition 6.38 that if the minimal representation of a continuous map $f : X \to Y$ is acyclic-valued, then any of its acyclic-valued representations may be used to compute its homology. Given the examples presented so far, one may be lulled into assuming that if the minimal representation is not acyclic-valued, then no representation is acyclic-valued. As the following example demonstrates, this is false. This in turn raises the question: if one has an acyclic-valued representation can it be used to compute the homology of f?

Example 6.55 Take X, f and λ as in Example 6.37. Let $Y := [0,1] \times [0,1]$ and consider $g : X \to Y$ given by the same formula as f, i.e. by

$$g(x_1, x_2) := \begin{cases} \lambda(x_2) & \text{if } (x_1, x_2) \in K_1 \cup K_3 \\ \lambda(x_1) & \text{if } (x_1, x_2) \in K_2 \cup K_4. \end{cases}$$

Then $M_f = M_g$, i.e. the minimal representation of g is not acyclic-valued. However, taking the cubical map $F : X \rightrightarrows Y$ given by

$$F(x_1, x_2) := [0,1] \times [0,1]$$

we easily see that F is an acyclic-valued representation of g. This gives an affirmative answer to the first question.

The answer to the other question is also positive, as the following proposition shows.

Proposition 6.56 *Assume $F : X \rightrightarrows Y$ is an acyclic-valued representation of a continuous map $f : X \to Y$. Then*

$$f_* = F_*.$$

Proof. Let α be a scaling such that M_{f^α} is acyclic-valued. Then for any $x \in X$ we have

$$\begin{aligned} M_{f^\alpha} \circ M_{\Lambda_X^\alpha}(x) &= \operatorname{ch}(f^\alpha(\operatorname{ch}(\Lambda_X^\alpha(\operatorname{ch}(x))))) \\ &= \operatorname{ch}(f^\alpha(\Lambda_X^\alpha(\operatorname{ch}(x)))) \\ &= \operatorname{ch}(f(\operatorname{ch}(x))) \\ &= M_f(x) \\ &\subset F(x). \end{aligned}$$

Therefore, by Proposition 6.30

$$f_* = (M_{f^\alpha})_* \circ (M_{\Lambda_X^\alpha})_* = F_*.$$

The final issue we need to deal with involves the composition of continuous functions. We will use the following technical lemma.

Lemma 6.57 *Let X and Y be cubical sets and let $f : X \to Y$ be continuous. Let α be a scaling vector. If M_f is acyclic-valued, then $M_{\Lambda^\alpha_Y} \circ M_f$ is acyclic-valued.*

Proof. Let $x \in X$. Observe that

$$M_{\Lambda^\alpha_Y} \circ M_f(x) = \mathrm{ch}\,(\Lambda^\alpha_Y(\mathrm{ch}\,(M_f(x)))) = \Lambda^\alpha_Y(M_f(x)).$$

Since $M_f(x)$ is acyclic-valued, it follows from Proposition 6.51 that $\Lambda^\alpha_Y(M_f(x))$ is also acyclic-valued. □

Theorem 6.58 *Assume $f : X \to Y$ and $g : Y \to Z$ are continuous maps between cubical sets. Then*

$$(g \circ f)_* = g_* \circ f_*$$

Proof. Let $h := g \circ f$. Select a scaling vector β such that M_{g^β} is acyclic-valued and for any $x, y \in Y^\beta$

$$\mathrm{dist}\,(x, y) \le 2 \quad \Rightarrow \quad \mathrm{dist}\,(g^\beta(x), g^\beta(y)) \le \frac{1}{2}. \tag{6.20}$$

Similarly, select a scaling vector α such that M_{f^α} and M_{h^α} are acyclic-valued, and for any x, y in X^α

$$\mathrm{dist}\,(x, y) \le 2 \quad \Rightarrow \quad \mathrm{dist}\,(\Lambda^\beta \circ f^\alpha(x), \Lambda^\beta \circ f^\alpha(y)) \le \frac{1}{2}. \tag{6.21}$$

Then the maps $\Lambda^\beta \circ f^\alpha$ and $g^\beta \circ (\Lambda^\beta \circ f^\alpha) = h^\alpha$ have acyclic-valued cubical representations. In particular, we can apply Lemma 6.57 to M_{f^α} and $M_{\Lambda^\beta_{X^\alpha}}$ which allows us to conclude from Lemma 6.41 that

$$(\Lambda^\beta_{X^\alpha} \circ f^\alpha)_* = \Lambda^\beta_{X^\alpha *} \circ f^\alpha_*. \tag{6.22}$$

Moreover, by (6.20) and (6.21), for any $x \in X^\alpha$

$$\mathrm{diam}\,(g^\beta \circ \mathrm{ch}\, \circ \Lambda^\beta \circ f^\alpha \circ \mathrm{ch}\,(x)) < 1.$$

Therefore, by Proposition 6.39

$$M_{g^\beta} \circ M_{\Lambda^\beta \circ f^\alpha}(x) = \mathrm{ch}\, \circ g^\beta \circ \mathrm{ch}\, \circ \Lambda^\beta \circ f^\alpha \circ \mathrm{ch}\,(x)$$

is acyclic, i.e. the composition $M_{g^\beta} \circ M_{\Lambda^\beta \circ f^\alpha}$ is acyclic-valued.

This implies that

$$h^\alpha_* = (g^\beta \circ \Lambda^\beta_{X^\alpha} \circ f^\alpha)_* = g^\beta_* \circ (\Lambda^\beta_{X^\alpha} \circ f^\alpha)_*. \tag{6.23}$$

Finally we get from (6.22) and (6.23)

$$\begin{aligned}(g \circ f)_* &= h_* \\ &= h_*^\alpha \circ (\Lambda_X^\alpha)_* \\ &= g_*^\beta \circ (\Lambda_{X^\alpha}^\beta \circ f^\alpha)_* \circ (\Lambda_X^\alpha)_* \\ &= g_*^\beta \circ (\Lambda_{X^\alpha}^\beta)_* \circ f_*^\alpha \circ (\Lambda_X^\alpha)_* \\ &= g_* \circ f_*.\end{aligned}$$

□

Because of the results of Section 2.3 we have a good understanding of the topological meaning of $H_0(X)$ for any cubical set X. Thus, it is natural to try to understand how the homology map acts on this homology group.

Proposition 6.59 *If $f : X \to X$ is a continuous map on a connected cubical set, then $f_* : H_0(X) \to H_0(X)$ is the identity map.*

Proof. Consider first the case when M_f is acyclic-valued. Then the homology map $f_* : H_0(X) \to H_0(X)$ is determined by an appropriate chain map $\varphi_0 : C_0(X) \to C_0(X)$. Which in turn can be determined by M_f acting on $\mathcal{K}_0(X)$. So let $Q \in \mathcal{K}_0(X)$. By definition $M_f(Q) = \operatorname{ch}(f(Q))$ which is an elementary cube. Let $P \in \mathcal{K}_0(\operatorname{ch}(f(Q)))$. Then, we can define $\varphi_0(\widehat{Q}) = \widehat{P}$, in which case $f_*([\widehat{Q}]) = [\widehat{P}]$. By Theorem 2.59, $[\widehat{Q}] = [\widehat{P}] \in H_0(X)$, and hence we have the identity map on $H_0(X)$.

When M_f ia not acyclic-valued, we have to go through the process of rescaling. We leave the details as an exercise. □

Exercises

6.9 Prove the conclusions announced in Example 6.43 about

(a) The immersion $j : X \to \{0\} \times X$ discussed in Example 4.6,
(b) The translation $t : X \to \mathbf{m} + X$ discussed in Example 4.7,
(b) The projection $p : Q \to Q$ discussed in Example 4.8.

6.10 Prove Proposition 6.45

6.11 Let X, λ, and f be as in Example 6.37.
a) Verify that the scaling by $\alpha := (2, 2)$ is sufficient for M_{f^α} to be a cubical representation of f^α.
b) Find a chain selector of M_{f^α}.
c) Compute the homology map of f. You may either compute it by hand or use the homology program for that.

6.12 Do the same as in Exercise 6.11 for the map given by

$$f(x_1, x_2) := \begin{cases} \lambda(x_2) & \text{if } (x_1, x_2) \in K_1 \\ \lambda(x_1) & \text{if } (x_1, x_2) \in K_2 \\ \lambda(1 - x_2) & \text{if } (x_1, x_2) \in K_3 \\ \lambda(1 - x_1) & \text{if } (x_1, x_2) \in K_4 \end{cases}$$

6.13 Continue the discussion in Example 6.27 in the context of rescalings to show that every continuous map $f : X \to X$ on a connected cubical set induces the identity map in 0-dimensional homology, namely

$$f_{*0} = \mathrm{id}_{H_0(X)}.$$

6.5 Homotopy Invariance

We now have a homology theory at our disposal. Given a cubical set X we can compute its homology groups $H_*(X)$ and given a continuous map f between cubical sets we can compute the induced map on homology f_*. What is missing is how these algebraic objects relate back to topology. Section 2.3 was a partial answer in that we showed that $H_0(X)$ counts the number of connected components of X. In this section we shall use homology to provide further classification of maps and spaces.

Definition 6.60 Let X and Y be cubical sets. Let $f, g : X \to Y$ be continuous functions. f is *homotopic* to g if there exists a continuous map $h : X \times [0,1] \to Y$ such that

$$h(x,0) = f(x) \quad \text{and} \quad h(x,1) = g(x)$$

for each $x \in X$. The map h is called a *homotopy* between f and g. The relation f homotopic to g is denoted by $f \sim g$.

It is left as Exercise 6.14 to check that homotopy is an equivalence relation. Homotopy of maps can next be used to define an equivalence between spaces:

Definition 6.61 Two cubical sets X and Y have the same *homotopy type* or, shortly, X and Y are *homotopic* if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \sim \mathrm{id}_X \quad \text{and} \quad f \circ g \sim \mathrm{id}_Y$$

where id_X and id_Y denote the identity maps. The relation X homotopic to Y is denoted by $X \sim Y$.

Example 6.62 Two cubical sets can appear to be quite different and still be homotopic. For example it is clear that the unit ball with respect to the supremum norm $B_0^d = [-1,1]^d$ is not homeomorphic to the point $\{0\}$. On the other hand these two sets are homotopic. To see this let $f : B_0^d \to \{0\}$ be defined by $f(x) = 0$ and let $g : \{0\} \to B_0^d$ be defined by $g(0) = 0$. Observe that $f \circ g = \mathrm{id}_{\{0\}}$ and hence $f \circ g \sim \mathrm{id}_{\{0\}}$. To show that $g \circ f \sim \mathrm{id}_{B_0^d}$ consider the map $h : B_0^d \times [0,1] \to B_0^d$ defined by

$$h(x,s) = (1-s)x.$$

Clearly, $h(x,0) = x = \mathrm{id}_{B_0^d}$ and $h(x,1) = 0$.

Notice that the homotopy constructed in the above example is also a deformation retraction (see Definition 2.70) of B_0^d onto $\{0\}$. We introduced the concept of deformation retraction for the purpose of elementary collapses. What is the relation between that concept and the concept of homotopy? We start from the following definition.

Definition 6.63 Given a pair $A \subset X$, a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ is called a *retraction.* If such a map exists, A is called a *retract* of X.

Note that, in particular, any retract of X must be closed in X. The notion of retraction is a direct analogy of projection of a vector space onto a subspace known from linear algebra with the attribute "linear" replaced by "continuous".

Definition 2.70 implies that a deformation retraction h is a homotopy between the identity map on X and the retraction $x \to h(x, 1)$ of X onto A. More precisely, we have the following result.

Theorem 6.64 *If A is a deformation retract of X then A and X have the same homotopy type.*

Proof. Let h be a deformation retraction of X to A. Define $f : X \to A$ by $f(x) := h(x, 1)$ and let $g : A \to X$ be the inclusion map. It follows from the properties of F that $f \circ g = \mathrm{id}_A$ and $g \circ f \sim \mathrm{id}_X$. □

It is easy to check that if A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

Definition 6.65 A cubical set X is *contractible* if the identity map on X is homotopic to a constant map, equivalently, if there exists a deformation retraction of X to a single point of X.

The relation between the contractibility and connectedness is illustrated in the following example.

Example 6.66 Any single vertex $\{v\}$ of any cubical set $X \subset \mathbf{R}^d$ is a retract of X with $r : X \to \{v\}$, $r(x) := v$. However $\{0\}$ is not a deformation retract of the two-point set $X = \{0, 1\}$. Indeed, if there was a deformation retraction h, the continuous map $s \to h(1, s)$ would send the connected interval $[0, 1]$ onto the disconnected set X, which would contradict Theorem 12.46.
By the same argument one may show that no disconnected space is contractible.

The next theorem provides the bridge from topology to algebra.

Theorem 6.67 *Let X and Y be cubical sets and let $f, g : X \to Y$ be homotopic maps. Then*

$$f_* = g_*.$$

Proof. First note that since X is cubical, $X \times [0,1]$ is also cubical and, for $k = 0, 1$ and $x \in X$, the maps $j_k : X \to X \times [0,1]$ given by

$$j_k(x) := (x, k)$$

are continuous. Let $h : X \times [0,1] \to Y$ be the homotopy joining f and g, i.e. a continuous function such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$, which may be written also as

$$\begin{aligned} f &= h \circ j_0, \\ g &= h \circ j_1. \end{aligned} \tag{6.24}$$

Let $\mathcal{F} : \mathcal{K}_{\max}(X) \rightrightarrows \mathcal{K}(X \times [0,1])$ be defined for $Q \in \mathcal{K}_{\max}(X)$ by

$$\mathcal{F}(Q) := \{Q \times [0,1]\}$$

and let $F := \lfloor \mathcal{F} \rfloor$. Then, by Theorem 6.14 and Corollary 6.19, F is a lower semicontinuous cubical map. One easily verifies that if $x \in X$ and $P \in \mathcal{K}(X)$ is the unique cube such that $x \in \overset{\circ}{P}$ then

$$F(x) = P \times [0,1].$$

Therefore, F is acyclic-valued. Since obviously $j_k(x) \in F(x)$, we see that F is an acyclic-valued representation of j_0 and j_1. Therefore, we have, by (6.24), Theorem 6.58 and Proposition 6.56,

$$f_* = h_* \circ (j_0)_* = h_* \circ F_* = h_* \circ (j_1)_* = g_*.$$

□

Corollary 6.68 *Let X and Y be homotopic cubical sets. Then*

$$H_*(X) \cong H_*(Y).$$

Proof. See Exercise 6.19. □

Exercises

6.14 Prove that homotopy is an equivalence relation.

6.15 Let X, Y be cubical sets and $f, g : X \to Y$ be continuous maps. Under the following assumptions on X and Y construct a homotopy $f \sim g$.

- $X = Y = [0,1]$
- $X = \Gamma^1$ and $Y = [0,1]$, where Γ^1 is the boundary of the unit square $Q = [0,1]^2$.
- X is any cubical set and $\{y\} \in Y$ is a deformation retract of Y.

6.16 A subset $C \subset \mathbf{R}^d$ is called *star-shaped* if there exists a point $x_0 \in C$ such that for any $y \in C$ the line segment $[x_0, y]$ is contained in C. A convex set (recall Definition 12.49) is a particular case of a star-shaped set. Prove that any star-shaped set is contractible.

6.17 Show that every tree is contractible to a vertex.

6.18 In one of the exercises of the previous section, either 6.11 or 6.12, you should have got the trivial homology map. Prove this in a different way, by explicitly defining a homotopy from f to a constant map.

6.19 Prove Corollary 6.68.

6.20 Let $\Pi := \{f : \Gamma^1 \to \Gamma^1 \mid f \text{ is continuous}\}$. Give a lower bound on the number of homotopy classes in Π.

7

Computing Homology of Maps

In Chapter 6 we provided a theoretical construction for producing a homology map $f_* : H_*(X) \to H_*(Y)$ given an arbitrary continuous function $f : X \to Y$ between two cubical sets. In this chapter we will provide algorithms that allow us to use the computer to obtain f_*.

There are four steps involved in computing the homology of a map. The first is to construct a function $\mathcal{F} : \mathcal{K}_{\max}(X) \to \operatorname{Rect}(\mathbf{R}^{d'})$ which assigns to every $Q \in \mathcal{K}_{\max}(X)$ a collection of elementary cubes $\mathcal{F}(Q)$ such that

$$f(Q) \subset |\mathcal{F}(Q)|, \tag{7.1}$$

where $|\mathcal{F}(Q)|$ is defined as in (2.10).

The second step is to use the map $\mathcal{F}$ to construct a multivalued lower semicontinuous representation $F : X \rightrightarrows Y$ of f by means of the formula

$$F(x) := \bigcap \{|\mathcal{F}(Q)| \mid x \in Q \in \mathcal{K}_{\max}(X)\} \tag{7.2}$$

(see Theorem 6.14 and Corollary 6.19). The third step is to compute a chain selector of F and the fourth and final step is to determine the homology map from the chain selector.

We begin our discussion with the first step. Observe that this is purely an approximation problem which has many classical numerical solutions and most of them can be adapted to our needs. However, given the goals of this book we do not want to divert too much from homology, so we only briefly describe one of the simplest approaches based on the class of rational maps already considered in Section 5.1. Though simple to represent, for example one can take just the list of the coefficients as a computer representation, actually it encompasses a reasonably large class of maps. As we explained in Section 5.1, interval arithmetic provides a simple means of solving the approximation problem in this case. Using formulas (5.1) we can easily overload arithmetic operators so that they can operate on intervals (compare Section 14.4 for operator overloading).

Since we identify a sequence of d intervals $I_1, I_2, \dots I_d$ with the rectangle $I_1 \times I_2 \times \cdots \times I_d$, the variables of data type

typedef `rectangle` := **array**[1 :] **of** `interval`

may be interpreted as storing either rectangles or just sequences of intervals. Without going into details we just assume that the coefficients of a rational fucntion are stored in the data type

typedef `rationalMap`;

Now, using the concepts of interval arithmetic and depending on the particular definition of the data structure `rationalMap`, it not too difficult to write an algorithm

function `evaluate`(`rationalMap f`, `rectangle R`)

which given a rational map f and a rectangle R returns a possibly small rectangle R' such that $f(R) \subset R'$.

For instance, if we restrict our attention to the simplest case of a polynomial of one variable with integer coefficients, then we can take

typedef `rationalMap` = **array**[0 :] **of int** ;

and then the algorithm `evaluate` may look as follows

```
function evaluate(rationalMap f, rectangle R)
interval s = 0, xp = 1;
for i := to lastIndex(f) do
    s := s + f[i] * xp;
    xp := xp * R[1];
endfor;
return s;
```

Extending this approach to a rational function of one variable is straightforward. Extending it to more variables is lengthy but not difficult. We leave it as an exercise. (See Exercises ...).

Once we have an appropriate data structure `rationalMap` and a respective algorithm `evaluate`, we can define $\mathcal{F}(Q)$ by

$$\mathcal{F}(Q) := \left\{ P \in \mathcal{K} \mid P \subset \hat{f}(Q) \right\}.$$

where $\hat{f}(Q)$ stands for the evaluation of f on Q in interval arithmetic, computed by the algorithm `evaluate`. However, since in this case $|\mathcal{F}(Q)|$ is exactly

$\hat{f}(Q)$, we can eliminate $\mathcal{F}$ from consideration and define the multivalued lower semicontinuous representation of f directly by

$$F(x) := \bigcap \left\{ \hat{f}(Q) \mid x \in Q \in \mathcal{K}_{\max}(X) \right\}. \tag{7.3}$$

This will be done in the next section. In Section 7.2 we provide the details of computing a chain selector of F and the fourth and final step in which we determine the homology map from the chain selector is discussed in Section 7.3

Exercises

7.1 Write algorithms overloading the operators "+", "-", "*" and "/" so that they can be used with variables of type `interval` in the sense of (5.1).

7.2 Design a respective data structure and write an algorithm which evaluates a rational function of one variable on an interval.

7.3 Design a respective data structure and write an algorithm which evaluates a polynomial of two variables on intervals.

7.4 ** Design a respective data structure and write an algorithm which evaluates a rational function of an arbitrary finite number of variables on intervals.

7.1 Producing Multivalued Representation

The goal of this section is to use formula (7.3) to produce a multivalued representation of the rational map f. For this end we need first an algorithm, which, given a cell $\overset{\circ}{Q}$, will

produce a list of all elementary cubes which have non-empty intersection with $\overset{\circ}{Q}$. This is done below, where it is assumed that the algorithm `degenerateDimensions` returns a list of the dimensions in which the respective elementary intervals are degenerate (see Exercise 7.5).

Algorithm 7.1 Neighbors of an Elementary Cell

```
function neighbors(cube Q)
L := degenerateDimensions(Q);
N := ∅;
for each K in subsets(L) do
    for each M in subsets(K) do
        R := Q;
        for each k in K do
            if k in M then
                R[k]{left} := Q[k]{left} − 1;
            else
                R[k]{right} := Q[k]{right} + 1;
```

```
            endif;
         endfor;
         N := join(N, R);
      endfor;
   endfor;
   return N;
```

Example 7.2 If $Q = [1,2] \times [3] \subset \mathbf{R}^2$, then the output of Algorithm 7.1 is $\{[1.2] \times [2,3], [1.2] \times [1,2]\}$.

Proposition 7.3 *Assume Algorithm 7.1 is called with* `Q` *containing the representation of an elementary cube* Q. *Then it stops and returns a list of the neighbors of* $\overset{\circ}{Q}$.

Proof. Obviously the algorithm always stops. Let

$$Q = I_1 \times I_2 \times \cdots I_m$$

be the decomposition of Q into the elementary intervals and put

$$L := \{i = 1, 2, \ldots m \mid \dim I_i = 0\}.$$

Let N denote the list of cubes returned by the algorithm and let

$$R = J_1 \times J_2 \times \cdots J_m$$

be one of the cubes added to the list. It follows from the construction of R that $J_i = I_i$ whenever $i \not\in L$ or J_i is degenerate and J_i is an elementary interval containing I_i in all other cases. Therefore R is an elementary cube and $J_i \cap \overset{\circ}{I}_i \neq \emptyset$ for every $i = 1, 2, \ldots m$. It follows that $R \cap \overset{\circ}{Q} \neq \emptyset$.

Assume in turn that R is an elementary cube satisfying $R \cap \overset{\circ}{Q} \neq \emptyset$. Then $J_i \cap \overset{\circ}{I}_i \neq \emptyset$ for every $i = 1, 2, \ldots m$, which means that either I_i is non-degenerate and $I_i = J_i$ or I_i is degenerate and $I_i \subset J_i$. Let

$$K := \{i \in L \mid \dim J_i = 1\}$$

and let M be the set of such $i \in K$ that the left endpoint of J_i is less that the left endpoints of I_i. It is straightforward to verify that the elementary cube R is added to the list N when the variable `K` takes value K and the variable `M` takes value M. □

For the next algorithm we need to describe first three elementary algorithms, whose details are left as an exercise (see Exercise 7.6). The algorithms are

```
function rectangularCover(cubicalSet X)
function rectangleIntersection(rectangle M, N)
```

function elementaryCubes(rectangle M, cubicalSet X)

We assume that the first algorithm accepts a cubical set and returns the smallest rectangle which contains the cubical set, the second just returns the rectangle which is the intersection of the two rectangles given as arguments and the third accepts a rectangle and a cubical set on input and returns a cubical set which is their intersection. Now we are ready to present an algorithm constructing a multivalued lowersemicontinuous representation of a rational map.

Algorithm 7.4 Multivalued Representation

```
function multivaluedRepresentation(cubicalSet X, Y, rationalMap f)
M := rectangularCover(Y);
E := cubicalChainGroup(X);
for i := 0 to lastIndex(E) do
    for each Q in E[i] do
        L := neighbors(Q);
        for each R in L do
            if R in X then
                fR := evaluate(f, R);
                M := rectangleIntersection(M, fR);
            endif;
        endfor;
        F[i]{Q} := elementaryCubes(M, Y);
    endfor;
endfor;
return F;
```

Theorem 7.5 *Assume Algorithm 7.4 is called with* X, Y *containing the representation of two cubical sets X, Y and* f *is a representation of a rational fucntion $f : X \to Y$. Then it stops and returns an array* F *containing a lower semicontinuous multivalued representation of f.*

Proof. Let F be the multivalued map represented by F. It is straightforward to verify that F satisfies formula (7.2). It follows from Theorem 6.14 and Corollary 6.19 that F is cubical and lower semicontinuous. It remains to prove (6.16) but this property follows from (5.2) and (**??**). □

Exercises

7.5 To complete Algorithm 7.1, write an algorithm degenerateDimensions, which given an elementary cube $Q = I_1 \times I_2 \times \cdots \times I_d$ returns the list of those i for which I_i is degenerate.

7.6 To complete Algorithm 7.4, write the three algorithms described in the discussion preceding the algorithm.

7.2 Chain Selector Algorithm

Our next step is the construction of chain selectors from lower semicontinuous multivalued maps. We begin with the following simple algorithm evaluating a chain map at a chain. In this algorithm we use the structure

typedef `chainMap` = **array**[0 :] **of hash**{`cube`} **of** `chain`;

to store chain maps. Note that the structure does not differ form the structure `boundaryMap`, which we use to store the boundary map.

Algorithm 7.6 Chain Evaluation

```
function evaluate(chainMap phi, int i, chain c)
chain d := ();
K := keys(c);
for each Q in K do
    d := d + c{Q} * phi[i]{Q};
endfor;
return d;
```

The following proposition is straightforward.

Proposition 7.7 *Assume Algorithm 7.6 is called with* ϕ *containing a chain map and* `c` *containing a chain. Then it stops and returns the value* $\phi($`c`$)$ *of the chain map* ϕ *on the chain* `c`*.*

Now we are ready to present the algorithm computing the chain selector. The input to this algorithm will use the following data structure.

typedef `multivaluedMap` = **array**[0 :] **of hash**{`cube`} **of** `cube`;

We will also use a straightforwad to write algorithm `selectVertex`, which, given an elementary cube returns one of its vertices.

Algorithm 7.8 Chain Selector

```
function chainSelector(multivaluedMap F)
chainMap phi := ();
K := keys(F[0]);
for each Q in K do
    phi[0]{Q} := selectVertex(F[0]{Q});
endfor;
for i := 1 to lastIndex(F) do
    K := keys(F[i]);
    for each Q in K do
        z := evaluate(phi, boundaryOperator(Q));
```

```
        phi[i]{Q} := preBoundary(z, F[i]{Q});
    endfor;
endfor;
return phi;
```

Proposition 7.9 *Assume Algorithm 7.8 is called with* F *containing an acyclic valued, representable lower semicontinuous map. Then it stops and returns a chain selector of* F*.*

Proof. The proposition follows easily from Theorem 6.22. □

7.3 Computing Homology of Maps

We begin with an algorithm, which given a cycle and a basis of a group of cycles will return the chain written as a linear combination of the elements of this basis. We assume the basis is stored in the data structure

```
typedef basis = list of chain;
```

Algorithm 7.10 Coordinates in a Given Basis

```
function coordinates(chain z, basis D)
list of cube canonicalBasis := ();
for each d in D do
    canonicalBasis := union(canonicalBasis, keys(d));
endfor;
matrix V;
for j := 1 to length(D);
    V[j] := canonicalCoordinates(D[j], canonicalBasis);
endfor;
xi := Solve(V, z);
return chainFromCanonicalCoordinates(xi, canonicalBasis);
```

Proposition 7.11 *Assume Algorithm 7.10 is called with* z *containing a cycle and* D *containing a list of basis vectors of a group of cycles. Then it stops and returns a hash containing the coordinates of* z *in the basis* D*.*

□

Now we are ready to present an algorithm computing the homology of a continuous map.

Algorithm 7.12 Homology of a Continuous Map

```
function homologyOfMap(cubicalSet X, Y, rationalMap f)
HX := homology(X);
HY := homology(Y);
F := multivaluedRepresentation(X, Y, f);
```

```
phi := chainSelector(F);
for k := 0 to lastIndex(HX[k]) do
    C := HX[k]{generators};
    D := HY[k]{generators};
    for each c in C do
        d := evaluate(phi, c);
        xi := coordinates(d, D);
        Hphi[k]{c} := xi;
    endfor;
endfor;
return (HX, HY, Hphi);
```

Theorem 7.13 *Assume Algorithm 7.12 is called with* X, Y *containing representations of two cubical sets* X, Y *and* f *containing a representation of a continuous map* $f : X \to Y$. *Then it stops and returns a triple* (HX, HY, Hphi) *such that* HX, HY *represent the homology of* X *and* Y *in the sense of Theorem 3.76 and* Hphi *is an array such that for every cycle* $c \in \mathbf{Z}_k(X)$ *whose representation* c *is on the list of generators* HX[k]{generators} *the value* $H_k(f)(c)$ *written in terms of the generators* HY[k]{generators} *is given by* Hphi[k]{c}.
□

7.4 Computing Homology Maps - Reduction Approach

7.4.1 Reduction of a Chain Map

Given a triple $(\mathcal{C}, \mathcal{D}, \varphi)$ where $\mathcal{C}$, $\mathcal{D}$ are finitely generated chain complexes and $\varphi : \mathcal{C} \to \mathcal{D}$ is a chain map, an elementary reduction performed either on C or D as described in the previous section, induces a reduction on the whole triple.

Indeed, if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by an elementary reduction, we define $\varphi' : \mathcal{C}' \to \mathcal{D}'$ by $\varphi' := \varphi\iota$, where $\iota : \mathcal{C}' \to \mathcal{C}$ is the inclusion chain map given in the proof of Theorem 4.23. If $\mathcal{D}'$ is obtained from $\mathcal{D}$, we define $\varphi' : \mathcal{C} \to \mathcal{D}'$ by $\varphi' := \pi'\varphi$ where $\pi' : \mathcal{D} \to \mathcal{D}'$ is the projection chain map given for $\mathcal{D}$ as in Corollary **??**. Since we showed in the proof of Theorem 4.23 that ι and π' induce isomorphisms in homology, this procedure can be iterated so that in the final stage we get a chain map $\varphi^f : C^f \to D^f$ such that the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{D} \\
\uparrow \iota^f & & \downarrow \pi^f \\
\mathcal{C}^f & \xrightarrow{\varphi^f} & \mathcal{D}^f
\end{array}$$

commutes and the vertical arrows induce isomorphisms in homology.

Now Corollary 4.29 and the above discussion imply the following

Corollary 7.14 *Suppose that the elementary reductions can be successfully performed until $\partial^f = 0$. Then the triple* $(\mathcal{C}^f, \mathcal{D}^f, \varphi^f)$ *is isomorphic to the triple* $(H(\mathcal{C}), H(\mathcal{D}), \varphi_*)$ *in the sense that the vertical arrows in the following commuting diagram are isomorphisms:*

$$\begin{array}{ccc} H(\mathcal{C}) & \xrightarrow{\varphi_*} & H(\mathcal{D}) \\ \uparrow \iota^f_* & & \downarrow \pi^f_* \\ \mathcal{C}^f & \xrightarrow{\varphi^f} & \mathcal{D}^f \end{array}$$

7.4.2 Computing Homology of a Chain map

We may return now to the topic discussed in Section 7.4.1 and describe an algorithm of computing mod p homology of a chain map $\varphi : \mathcal{C} \to \mathcal{D}$. For each $k \in \mathbf{Z}$ let A_k be the matrix of $\varphi_k : C_k \to D_k$ with respect to fixed bases W_k and U_k in C_k and D_k, respectively.

Given the reduction of elements a and b in C_{m-1} and C_m we define the matrix A'_k of $\varphi'_k : C'_k \to D_k$, for each k, by specifying its columns, using the map ι discussed in Section 7.4.1. If $k \neq m$ then $A'_k = A_k$. Assume then $k = m$. Let Col_w be the column of A_k which corresponds to $w \in W_m$, let $\lambda = \langle \partial b, a \rangle \neq 0$, and let $\alpha_w = \langle \partial w, a \rangle$, $w \in W'_m = W_m \setminus \{b\}$.

If $\alpha_w \neq 0$, put $\mathrm{Col}'_w = \mathrm{Col}_w$, otherwise $\mathrm{Col}'_w = \mathrm{Col}_w - \alpha_w \lambda^{-1} \mathrm{Col}_b$.

Analogically, given the reduction of elements a and b in D_{m-1}, D_m we define the matrix A'_k of $\varphi'_k : C_k \to D'_k$, for each k, by specifying its rows, using the definition of the map p' from discussed in Section 7.4.1.

If $k \neq m-1, m$ then $A'_k = A_k$. If $k = m$ we obtain A'_k by deleting the row Row_b which corresponds to b in A_k. Assume then $k = m-1$ and let $u \in U'_{m-1} = U_{m-1} \setminus \{a\}$. Let Row_u be the row of A_{m-1} which corresponds to u. Let Row_a be the row which corresponds to a and let $\beta_u = \langle \partial b, u \rangle$. It is easy to check that the row in the matrix A'_{m-1} which corresponds to u is given by

$$\mathrm{Row}'_u = \mathrm{Row}_u - \lambda^{-1} \beta_u \mathrm{Row}_a$$

For a fixed k the computation of $\varphi'_k : \mathcal{C}' \to \mathcal{D}'$ consists of two iterated transformations of the matrix A_k, each one taking $O(n^2)$ time. We can iterate the above matrix transformations until there is no possibility for a one-step reduction, finally obtaining the homomorphism $\varphi^* : H(\mathcal{C}; \mathbf{Z}_p) \to H(\mathcal{D}; \mathbf{Z}_p)$ corresponding to the chain map $\varphi : \mathcal{C} \to \mathcal{D}$. Since the dimension of the complexes is fixed we get an algorithm for computing the homology of a chain map of complexity $O(n^3)$.

7.5 Geometric Preboundary Algorithm

In Section 3.7 we gave an algebraic algorithm solving the equation

$$\partial c = z \tag{7.4}$$

for c having given a reduced cycle z in an acyclic cubical set X. A solution c was called a preboundary of z. That algorithm was, in turn, used for computing chain selectors in this chapter. As we mentioned earlier, the manipulation with large matrices in the algebraic preboundary algorithm due to a possibly very large number of cubes generating the groups $C_{m-1}(X)$ and $C_m(X)$ may lead to heavy calculations. We are therefore interested in searching for algorithms with a more geometric flavor which could be more efficient in particular cases of our interest.

We shall present here a geometric algorithm due to [4] which applies to reduced cycles z in a cubical rectangle $R \subset \mathbf{R}^d$. The algorithm is based on the recurrence with respect to the dimension of R.

Here is the main idea of the recurrence step. Let R' be a $(d-1)$-dimensional face of R. The projection of R onto R' induces a projection $\pi : \mathcal{C}(R) \to \mathcal{C}(R)$ with the image $\mathcal{C}(R')$ which will be defined the way it was done for the unit cube in Example 4.8. We let $z' := \pi z$ and compute $\mathrm{Pre}\,(z')$. Then $\mathrm{Pre}\,(z)$ is obtained by adding to $\mathrm{Pre}\,(z')$ "ribs" or "bands" of all $(k+1)$-dimensional unitary cubes through which z is projected, with appropriately chosen coefficients. We will show that our ribs are related to a chain homotopy between the projection chain map and identity constructed as in Example 4.11.

We start from a simple example.

Example 7.15 Consider the 2-dimensional rectangle $R = [0, 100] \times [0, 1]$ and consider the reduced 0-cycle $z = \widehat{V_1} - \widehat{V_0}$ where $V_1 = [100] \times [1]$ and $V_0 = [0] \times [0]$.

When projecting R onto $R' = [0] \times [0, 1]$, we get a new reduced cycle $z' = \widehat{V_1'} - \widehat{V_0}$ where $V_1' = [0] \times [1]$. Moreover V_1 is projected to V_1' through the interval $[0, 100] \times [1]$. We associate with this interval a "rib" defined as the sum of duals of all elementary intervals passed on the way:

$$\mathrm{Rib}\,(V_1) := (\sum_{i=1}^{100} \widehat{[i-1, i]}) \diamond \widehat{[1]} \tag{7.5}$$

It is easy to check that $\partial_1 \mathrm{Rib}\,(V_1) = \widehat{V_1} - \widehat{V_1'}$. The new cycle z' lies in the rectangle R' of dimension 1, so we reduced the dimension. We will show that

$$\mathrm{Pre}\,(z) := \mathrm{Pre}\,(z') + \mathrm{Rib}\,(V_1)$$

gives the right formula. The next projection of R' onto V_0 (which is, in this case, an elementary collapse) induces the projection of V_1' to V_0 along the interval $[0] \times [0, 1]$. The corresponding rib is $\mathrm{Rib}\,(V_1') := \widehat{[0]} \diamond \widehat{[0, 1]}$. Since the projected cycle is $z'' = \widehat{V_0} - \widehat{V_0} = 0$, we put $\mathrm{Pre}\,(z') := \mathrm{Rib}\,(V_1')$. We get

$$\partial_1 \mathrm{Pre}\,(z) = \widehat{V_1'} - \widehat{V_0} + \widehat{V_1} - \widehat{V_1'} = z$$

What permitted us writing the formula (7.5) without passing through those stages was the geometric intuition telling that the boundary of $[0, 100]$ is $\{0, 100\}$. We display an algebraic analogy of that simple fact by introducing the following notation for any pair of integers $k < l$:

$$\overrightarrow{[k, l]} := \sum_{i=k+1}^{l} \widehat{[i-1, i]}.$$

It is clear that $\overrightarrow{[k, l]} = \widehat{[k, l]}$ if $l = k + 1$ and an instant verification shows that

$$\partial \overrightarrow{[k, l]} = \widehat{[l]} - \widehat{[k]}.$$

With this notation, the formula (7.5) becomes transparent:

$$\mathrm{Rib}\,(V_1) := \overrightarrow{[0, 100]} \diamond \widehat{[1]}$$

and

$$\mathrm{Pre}\,(z) = \widehat{[0]} \diamond \widehat{[0, 1]} + \overrightarrow{[0, 100]} \diamond \widehat{[1]}$$

When using this notation we must be aware of the non-uniqueness of such expressions as

$$\overrightarrow{[0, 100]} - \overrightarrow{[50, 150]} = \overrightarrow{[0, 49]} - \overrightarrow{[101, 150]}\ .$$

Before we start the main construction, we need to generalize Examples 4.8 and 4.11 to arbitrary cubical rectangles. Let R be a cubical rectangle of the form

$$R = [m_1, M_1] \times [m_2, M_2] \times \ldots \times [m_d, M_d]$$

of dimension greater than 0 and let j_1 be the first non-degenerate coordinate of R, i.e. the smallest integer j with the property $m_j \neq M_j$. Consider now the rectangle

$$R' = \{x \in R \mid x_{j_1} = m_{j_1}\}$$

perpendicular to the first non trivial direction j_1 of R. Let $p : R \to R$ be the projection with the image R', defined coordinate-wise by

$$p(x)_i := \begin{cases} x_i & \text{if } i \neq j_1, \\ m_{j_1} & \text{otherwise.} \end{cases} \tag{7.6}$$

We denote by $p' : R \to R'$ the associated projection given by the same formula but with the target space R'. The two maps are related by the identity $p = ip'$ where $i : R' \hookrightarrow R$ is the inclusion map. For integer $k = 0, 1, \ldots \dim(R)$ consider an elementary cube $Q \in K_k(R)$. It can be expressed as the product of elementary intervals $Q = Q_1 \times Q_2 \times \cdots \times Q_d$. We have two cases: either Q_{j_1} is non-degenerated, say $Q_{j_1} = [a_{j_1}, a_{j_1} + 1]$ or it is degenerated, say $Q_{j_1} = [a_{j_1}]$. The projection p maps Q in both cases to an elementary cube

$$p(Q) = Q_1 \times \cdots Q_{j_1-1} \times [m_{j_1}] \times \mathbf{Q}_{j_1+1} \times \cdots \mathbf{Q}_d.$$

In the first case, this gives an elementary cube of dimension $k-1$. But, in the second case, $p(Q)$ is an elementary cube of the same dimension k.

This permits us to associate with p homomorphisms $\pi_k : C_k(R) \to C_k(R)$ on the level of chain groups defined on generators as follows.

$$\pi_k(\widehat{Q}) := \begin{cases} \widehat{p(Q)} = \widehat{Q_1} \diamond \cdots \diamond \widehat{Q_{j_1-1}} \diamond \widehat{[m_{j_1}]} \diamond \widehat{Q_{j_1+1}} \cdots \diamond \widehat{Q_d} \\ \qquad\qquad \text{if } Q_{j_1} = [a_{j_1}], \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{7.7}$$

Note that $\pi_k(\widehat{Q}) = \widehat{Q}$ if $Q \subset R'$ hence π is a linear projection onto $\mathcal{C}(R')$. Here is an analogy of 4.19:

Lemma 7.16 *The collection of projections $\pi = \{\pi_k\} : \mathcal{C}(R) \to \mathcal{C}(R)$ is a chain map. In particular π maps cycles to cycles.*

Proof. Let $Q \in \mathcal{K}_k(R)$. We need to show that $\partial\pi\widehat{Q} = \pi\partial\widehat{Q}$. Since j_1 is the first nontrivial coordinate of R, Q can be decomposed as $Q = V \times Q_{j_1} \times P$ where $\dim(V) = 0$ and either $Q_{j_1} = [a_{j_1}]$ or $Q_{j_1} = [a_{j_1}, a_{j_1}+1]$. By using the formula (2.6) we obtain

$$\partial\widehat{Q} = 0 + \widehat{V} \diamond \partial\widehat{Q_{j_1}} \diamond \widehat{P} + (-1)^{\dim(Q_{j_1})}\widehat{V} \diamond \widehat{Q_{j_1}} \diamond \partial\widehat{P}.$$

If $Q_{j_1} = [a_{j_1}]$ then $\partial\widehat{Q_{j_1}} = 0$ so we get

$$\pi\partial\widehat{Q} = \pi(\widehat{V} \diamond \widehat{Q_{j_1}} \diamond \partial\widehat{P}) = \widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \partial\widehat{P}$$

and also

$$\partial\pi\widehat{Q} = \partial(\widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \widehat{P}) = \widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \partial\widehat{P} = \pi\partial\widehat{Q}.$$

If $Q_{j_1} = [a_{j_1}, a_{j_1}+1]$ then $\pi\widehat{Q} = 0$ by definition, so $\partial\pi\widehat{Q} = 0$. On the other hand

$$\begin{aligned} \pi\partial\widehat{Q} &= \pi(\widehat{V} \diamond (\widehat{[a_{j_1}+1]} - \widehat{[a_{j_1}]}) \diamond \widehat{P} - \widehat{V} \diamond \widehat{Q_{j_1}} \diamond \widehat{\partial P}) \\ &= \widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \widehat{P} - \widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \widehat{P} - 0 \\ &= 0. \end{aligned}$$

□

We will later see that our chain map π is in fact induced by p in the sense which will be given in Chapter 6. We denote by $\pi' : \mathcal{C}(R) \to \mathcal{C}(R')$ the associated projection given by the same formula but with the target space $\mathcal{C}(R')$. The two maps are related by the identity $\pi = \iota\pi'$ where $\iota : \mathcal{C}(R') \hookrightarrow \mathcal{C}(R)$ is the inclusion chain map.

Next, given any $Q \in \mathcal{K}_k(R)$ as discussed above, define

$$\operatorname{Rib}(Q) := \begin{cases} \widehat{Q_1} \diamond \cdots \diamond \widehat{Q_{j_1-1}} \diamond \overrightarrow{[m_{j_1}, q_{j_1}]} \diamond \widehat{Q_{j_1+1}} \diamond \widehat{Q_d} \\ \qquad\qquad \text{if } Q_{j_1} = [a_{j_1}] \text{ and } a_{j_1} > m_{j_1}, \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{7.8}$$

Note that, by definition, $\operatorname{Rib}(Q)$ is a $(k+1)$-chain in R. If it is not trivial then its support is the $(k+1)$-dimensional rectangle through which Q is projected to $p(Q)$. With a little knowledge of anatomy we may observe that ribs add up to a chest. Let us define maps $\operatorname{Chest}_k : C_k(R) \to \mathbf{C}_{k+1}(R)$ for each k by the formula

$$\operatorname{Chest}_k(c) := \sum_{Q \in \mathcal{K}_k(R)} \langle \widehat{Q}, c \rangle \operatorname{Rib}(Q). \tag{7.9}$$

Note that each $c \in C_k(R)$ can be presented as

$$c = \sum_{Q \in \mathcal{K}_k(R)} \langle c, \widehat{Q} \rangle \widehat{Q}.$$

so the above formula defines a linear map. The reader may guess now that this map is a chain homotopy: we postpone the proof of it until the end of our construction.

Let now $z \in Z_k(R)$ be a k-cycle (reduced cycle if $k = 0$) in a cubical rectangle R. We construct, by induction with respect to the dimension n of R, a $(k+1)$-chain $\operatorname{Pre}(z)$. Obviously the dimension of R is at least k so the induction starts from $n = k$.

If $n = k$, we must have $z = 0$ so we may put

$$\operatorname{Pre}(z) := 0.$$

Indeed, in this case, $C_{k+1}(R) = 0$ so $B_k(R) = 0$ and $\tilde{Z}_k(R) = 0$ since $\tilde{H}_k(R) = 0$. Knowing that $z \in \tilde{Z}_k(R)$, it follows that $z = 0$.

Suppose the construction is done for dimensions up to a certain $n \geq k$ and let now $\dim(R) = n + 1$.

By Lemma 7.16, $\pi(z)$ is a k-cycle contained in R', which has dimension n. Hence the $(k+1)$-chain $\operatorname{Pre}(\pi(z))$ is well defined and

$$\partial_{k+1} \operatorname{Pre}(\pi_k(z)) = \pi_k(z)$$

by the induction hypothesis.

We define

$$\operatorname{Pre}(z) := \operatorname{Pre}(\pi(z)) + \operatorname{Chest}_k(z). \tag{7.10}$$

The property $\partial \operatorname{Pre}(z) = z$ will follow once we prove the following Lemma.

Lemma 7.17 *The collection of maps* $\operatorname{Chest}_k : C_k(R) \to \mathbf{C}_{k+1}(R)$ *is the chain homotopy between* π *and* $\operatorname{id}_{\mathcal{C}(R)}$. *As a consequence, if* $z \in Z_k(R)$, *then*

$$\partial_{k+1}(\operatorname{Chest}_k(z)) = z - \pi_k(z). \tag{7.11}$$

Proof. We use the same arguments as those used in Example 4.11. We should verify that condition (4.7) holds for $\psi = \pi$ and $\varphi = \mathrm{id}$. This means that for any $Q \in \mathcal{K}_k(R)$ we should have

$$\partial_{k+1}\mathrm{Chest}_k Q + \mathrm{Chest}_{k-1}\partial_k \widehat{Q} = \widehat{Q} - \pi\widehat{Q}\ . \tag{7.12}$$

We decompose Q as in the proof of Lemma 7.16 as $Q = V \times Q_{j_1} \times P$ where $\dim(V) = 0$ and either $Q_{j_1} = [a_{j_1}]$ or $Q_{j_1} = [a_{j_1}, a_{j_1}+1]$. Consider the case $Q_1 = [a_{j_1}]$ where $a_{j_1} > m_{j_1}$. Then

$$\begin{aligned}
&\partial_{k+1}\mathrm{Chest}_k Q + \mathrm{Chest}_{k-1}\partial_k \widehat{Q}\\
&= \partial_{k+1}(\widehat{V}\diamond \overrightarrow{[m_{j_1}, a_{j_1}]} \diamond\widehat{P}) + \mathrm{Chest}_{k-1}(\widehat{[a_{j_1}]} \diamond \partial_k \widehat{P})\\
&= \widehat{V} \diamond \widehat{[a_{j_1}]} \diamond \widehat{P} - \widehat{V} \diamond \widehat{[m_{j_1}]} \diamond \widehat{P}\\
&\quad -\widehat{V}\diamond \overrightarrow{[m_{j_1}, a_{j_1}]} \diamond\partial_k \widehat{P} + \widehat{V}\diamond \overrightarrow{[m_{j_1}, a_{j_1}]} \diamond\partial_k \widehat{P}\\
&= \widehat{Q} - \pi\widehat{Q}.
\end{aligned}$$

Next let $Q_1 = [m_{j_1}]$. Then each term in the left hand side of Equation (7.12) is zero by the definition of Chest_k and the right hand side is zero because $\widehat{\pi} Q = \widehat{Q}$. Finally let $Q_1 = [m_{j_1}, a_{j_1}]$ where $m_{j_1} < a_{j_1}$. Then $\mathrm{Chest}_k \widehat{Q} = 0$ so we get

$$\begin{aligned}
&\partial_{k+1}\mathrm{Chest}_k Q + \mathrm{Chest}_{k-1}\partial_k \widehat{Q}\\
&= 0 + \mathrm{Chest}_{k-1}(\widehat{V} \diamond [a_{j_1}] \diamond \widehat{P} - \widehat{V \diamond [m_{j_1}]} \diamond \widehat{P} - \widehat{V}\diamond \overrightarrow{[m_{j_1}, a_{j_1}]} \diamond\partial_k \widehat{P})\\
&= \widehat{V}\diamond \overrightarrow{[m_{j_1}, a_{j_1}]} \diamond\widehat{P} - 0 - 0 = \widehat{Q} = \widehat{Q} - \pi\widehat{Q}
\end{aligned}$$

because $\pi\widehat{Q} = 0$. This proves that Chest is a chain homotopy. Finally, if z is a cycle, then $\mathrm{Chest}_{k-1}\partial_k z = \mathrm{Chest}_{k-1} 0 = 0$ so we obtain Equation (7.11). □

Example 7.18 Consider the cycle

$$z := \widehat{E_8} + \widehat{E_7} + \widehat{E_6} + \widehat{E_5} - \widehat{E_4} - \widehat{E_3} - \widehat{E_2} - \widehat{E_1}$$

with edges

$$\begin{aligned}
&E_1 := [0, 1] \times [0],\ E_2 := [0, 2] \times [0],\ E_3 := [2] \times [0, 1],\\
&E_4 := [2] \times [1, 2],\ E_5 := [1, 2] \times [2],\ E_6 := [1] \times [1, 2]\\
&E_7 := [0, 1] \times [1],\ E_8 := [0] \times [0, 1].
\end{aligned}$$

The cycle z represents the clockwise oriented contour in $\mathbf{R}^2$ indicated on Figure 7.1. The smallest rectangle containing $|z|$ is $R = [0, 2]^2$ and the projection p sends it to $R' = [0] \times [0, 2]$. We get

$$\pi(\widehat{E_1}) = \pi(\widehat{E_2}) = \pi(\widehat{E_7}) = \pi(\widehat{E_5}) = 0$$

because those edges project to vertices of R. Next

$$\pi(\widehat{E_8}) = \pi(\widehat{E_3}) = \widehat{E_8}\ ,\ \pi(\widehat{E_6}) = \pi(\widehat{E_4}) = \widehat{[0] \times [1,2]},$$

so we get $z' := \pi(z) = 0$. From the above calculus we get

$$\mathrm{Rib}\,(E_1) = \mathrm{Rib}\,(E_2) = \mathrm{Rib}\,(E_7) = \mathrm{Rib}\,(E_5) = 0$$

and $\mathrm{Rib}\,(E_8) = 0$ because $E_8 \subset R'$. The only nontrivial ribs are

$$\mathrm{Rib}\,(E_3) = \widehat{Q_1} + \widehat{Q_2}\ ,\ \mathrm{Rib}\,(E_4) = \widehat{Q_3} + \widehat{Q_4}\ ,\ \mathrm{Rib}\,(E_6) = \widehat{Q_4},$$

where Q_1, Q_2, Q_3, and Q_4 are the squares indicated on Figure 7.1.

We get

$$\begin{aligned}\mathrm{Pre}\,(z) &= \mathrm{Pre}\,(z') - \mathrm{Rib}\,(E_3) - \mathrm{Rib}\,(E_4) + \mathrm{Rib}\,(E_6)\\ &= 0 - \widehat{Q_1} - \widehat{Q_2} - \widehat{Q_3} - \widehat{Q_4} + \widehat{Q_4}\\ &= -(\widehat{Q_1} + \widehat{Q_2} + \widehat{Q_3}).\end{aligned}$$

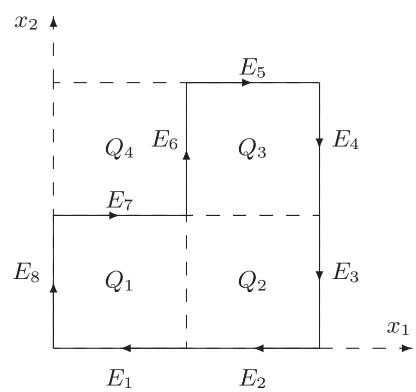


Fig. 7.1. Cycle $z = \widehat{E_8} + \widehat{E_7} + \widehat{E_6} + \widehat{E_5} - \widehat{E_4} - \widehat{E_3} - \widehat{E_2} - \widehat{E_1}$.

Example 7.19 Consider the 1-cycle $z = -\widehat{E_1} - \widehat{E_2} + \widehat{E_3} + \widehat{E_4} + \widehat{E_5} - \widehat{E_6}$ in $\mathbf{R}^3$ with edges

$$\begin{aligned}&E_1 := [1] \times [0] \times [0,1],\ E_2 := [0,1] \times [0] \times [0],\ E_3 := [0] \times [0,1] \times [0],\\ &E_4 := [0] \times [1] \times [0,1],\ E_5 := [0,1] \times [1] \times [1],\ E_6 := [1] \times [0,1] \times [1]\end{aligned}$$

The cycle z represents the contour on the surface of the unit cube $[0,1]^3$ indicated on Figure 7.2. The smallest rectangle containing $|z|$ is $R = [0,1]^3$ and we have $R' = [0] \times [0,1] \times [0,1]$. We get

$$\pi(\widehat{E_2}) = \pi(\widehat{E_5}) = 0$$

because those edges project to vertices of R'. Next

$$\pi(\widehat{E_1}) = [0] \times \widehat{[0] \times} [0,1] \ , \ \pi(\widehat{E_3}) = \widehat{E_3},$$
$$\pi(\widehat{E_4}) = \widehat{E_4} \ , \ \pi(\widehat{E_6}) = [0] \times \widehat{[0,1]} \times [1],$$

so we get

$$\pi(z) = -[0] \times \widehat{[0] \times} [0,1] + \widehat{E_3} + \widehat{E_4} - [0] \times \widehat{[0] \times} [0,1].$$

which represents an oriented contour of the square $Q' := [0] \times [0,1] \times [0,1]$. We leave the verification of the formula

$$\mathrm{Pre}\,(z') = \widehat{Q'}$$

as exercise.

By the same arguments as in the previous example we get

$$\mathrm{Rib}\,(E_2) = \mathrm{Rib}\,(E_3) = \mathrm{Rib}\,(E_4) = \mathrm{Rib}\,(E_5) = 0$$

The only nontrivial ribs are

$$\mathrm{Rib}\,(E_1) = \widehat{Q_1} := [0,1] \times \widehat{[0]} \times [0,1] \ , \ \mathrm{Rib}\,(E_6) = \widehat{Q_6} := [0,1] \times \widehat{[0,1]} \times [1].$$

The formula now gives

$$\begin{aligned} \mathrm{Pre}\,(z) &= \mathrm{Pre}\,(z') - \mathrm{Rib}\,(E_1) - \mathrm{Rib}\,(E_6) \\ &= \widehat{Q'} - \widehat{Q_1} - \widehat{Q_6}. \end{aligned}$$

We can see that the square Q' represents the left face of the cube on Figure 7.2, Q_1 represents its bottom face, and Q_6 represents its front face.

Remark 7.20 The preboundary construction is presented for one fixed rectangle R but there may be many cubical rectangles R containing $|z|$ and it might be advantageous to vary them. In that case, we will add the variable R as an index to expressions for the previously defined functions, e.g.

$$\pi_R(c) \ , \ \mathrm{Rib}_{\,R}(Q) \ , \ \mathrm{Chest}_{\,R}(z) \ ,$$

In particular, we may want to replace R in our previous discussion by $\mathrm{Rect}\,(z)$ defined as the smallest rectangle containing the support of z. It is easily seen that

$$\mathrm{Rect}\,(z) = [m_1, M_1] \times [m_2, M_2] \times \ldots \times [m_d, M_d] \tag{7.13}$$

where $m_i = \min \bigcup \{Q_i \mid \langle z, \widehat{Q} \rangle \neq 0\}$ and $M_i = \max \bigcup \{Q_i \mid \langle z, \widehat{Q} \rangle \neq 0\}$. Note that if we put $R = \mathrm{Rect}\,(z)$ and $R' = \pi_R(R)$ then R' might not be equal to $\mathrm{Rect}\,(\pi_R(z))$, we only know that $\mathrm{Rect}\,(\pi_R(z)) \subset R'$. In Example 7.18 we had $\pi(z) = 0$ so $\mathrm{Rect}\,(z) = \emptyset$. By passing from R' to $\mathrm{Rect}\,(\pi_R(z))$ after each projection, we may sometimes considerably speed up the construction of $\mathrm{Pre}\,(z)$.

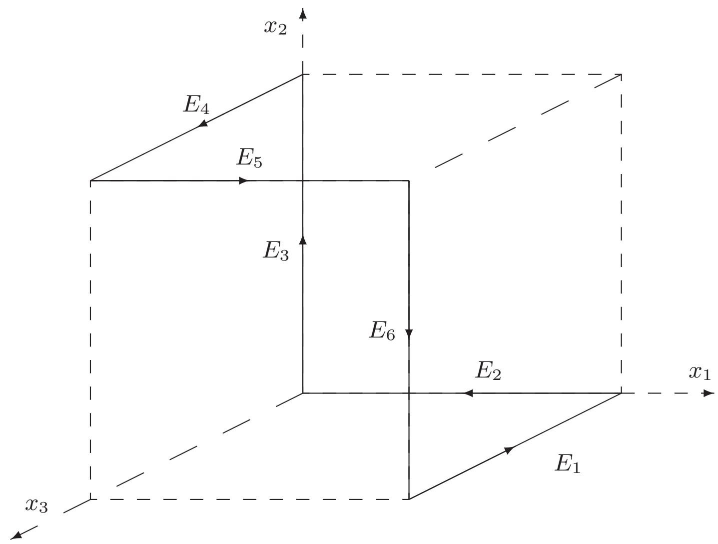


Fig. 7.2. Cycle $z = -\widehat{E_1} - \widehat{E_2} + \widehat{E_3} + \widehat{E_4} + \widehat{E_5} - \widehat{E_6}$.

Remark 7.21 It is visible from Figure 7.2 that different coboundaries of z may be obtained by projecting z to different faces of the cube. However, our formulas (7.7) and (7.10) are dependent on the choice of x_{j_1} as the first non-degenerate coordinate of R. If a different coordinate is chosen then a delicate analysis of sign adjustment is required due to the alternating sign in the formula for the boundary map.

We shall now present the formal algorithm based on the above preboundary construction. Our algorithm will use several functions. Since they are defined by explicit formulas, we leave algorithms computing them as exercises. First, the formula (7.13) defines the function `minRectangle` with variables of type `chain` (these are cycles) and values of type `cubicalSet` (these are rectangles). The formula (7.7) extended by linearity to chains gives rise to the function `projection` with variables and values of type `chain`. The formula (7.8) defines the function `rib` with variables of type `cube` and values of type `chain`. Finally, the function `rib` and the formula (7.9) give rise to the function `chest` with variables of type `chain chain`. We also leave the algorithm overloading the addition operator "+" for chains as an exercise.

Algorithm 7.22 Cubical Preboundary

function `cuPreBoundary(chain z)`
`R` := `minRectangle(z)`;
`n` := dim(`R`);
`k` := dim(`z`);
if $k = n$ **then**

```
        return 0;
    else
        z̄ := projection(z);
        c := chest(z);
        return cuPreBoundary(z̄) + c;
    endif;
```

Exercises

7.7 ** Consider a projection $p : R \to R'$ obtained by replacing the first non degenerate coordinate j_1 in the preboundary construction by the last non degenerate coordinate j_n, where $n = \dim(R)$. Obtain a version of the formula (7.10) for that projection.

7.8 ** Let $X = A \cup B$ where $A = [0,2] \times [0,1]$ and $B = [0,1] \times [0,2]$ and let $Y = A \cap B = [0,1]^2$ Construct, by analogy to the projection on the rectangle, a chain projection $\pi : \mathcal{C}(X) \to \mathcal{C}(X)$ with the image $\mathcal{C}(Y)$. Construct the chain homotopy between π and $\mathrm{id}_{\mathcal{C}(X)}$ and use it to carry over the geometric preboundary construction to cycles in X. Note that X is a simple example of a star-shape set.

7.9 Present algorithms for computing the functions used in Algorithm 7.22, namely

(a) `minRectangle`,
(b) `projection`,
(c) The addition of chains of the same dimension,
(d) `rib`,
(e) `chest`.

8

Prospects in Digital Image Processing

It is hard to think of a scientific or engineering discipline which does not generate computational simulations or make use of recording devices or sensors to produce or collect image data. It is trivial to record simple color videos of events, but it should be noted that such a recording can easily require about 25 megabytes of data per second. While obviously more difficult, it is possible, using X-ray computed tomography, to visualize cardiovascular tissue with a resolution on the order of 10 μm. Because this can be done at a high speed, timed sequences of 3-dimensional images can be constructed. This technique can be used to obtain detailed information about the geometry and function of the heart, but this entails large amounts of data. Obviously, the size and complexity of this data will grow as the sophistication of the sensors or simulations increase.

These large amounts of data are a mixed blessing; while we can be more confident that the desired information is captured, extracting the relevant information in a sea of data can become more difficult. One solution is to develop automated methods for image processing. These techniques are often, if somewhat artificially, separated into two categories *low-level vision* and *high level vision*. Typical examples of the latter include object recognition, optical character and handwriting recognizers and robotic control by visual feedback. Low level vision, on the other hand, focusses on the geometric structures of the objects being imaged. As such it often is a first step towards higher level tasks. The purpose of this chapter is to convince the reader that computational homology has the potential to play an important role in low level vision.

Notice the phrasing of the last sentence. The use of algebraic topology in imaging is, at the time that this book is being written, a very new subject.

8.1 Digital Images and Cubical Complexes

We began Chapter 2 by commenting on the apparent similarities between digital images consisting of pixels and cubical complexes. We now want to take a closer look at this analogy.

Consider the photo in Figure 8.1 of the moon's surface in the Sea of Tranquility taken from the Apollo 10 spacecraft. This is a black and white photo which if rendered on a computer screen would be presented as a rectangular set of elements each one of which is called a picture element, or a *pixel* for short. Each pixel has a specific light intensity determined by an integer grey scale value between 0 and 255. This rendering captures the essential elements of a digital image: the image must be defined on a discretized domain (the array of pixels) and similarly the observed values of the image must lie in a discrete set (grey scales values).

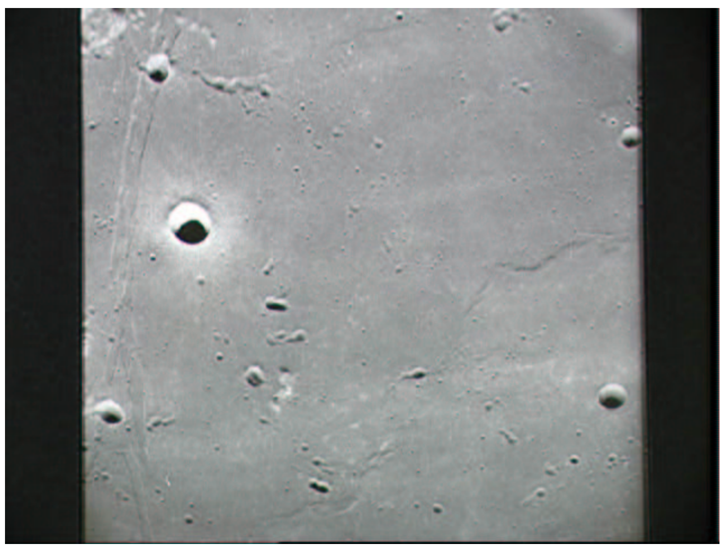


Fig. 8.1. Near vertical photograph taken from the Apollo 10 Command and Service Modules shows features typical of the Sea of Tranquility near Apollo Landing Site 2. The original is a 70mm black and white photo.

On the other hand, the Sea of Tranquility is an analog rather than a digital object. Its physical presence is continuous in space and time. Furthermore, the visual intensity of the image that we would see if we were observing the moon directly also takes a continuum of values. Clearly, information is being lost in the process of describing an analog object in terms of digital information. This is an important point and is the focus of considerable research in the image processing community. However, these problems lie outside the scope of this book and with the exception of the following example and occasional comments will not be discussed further.

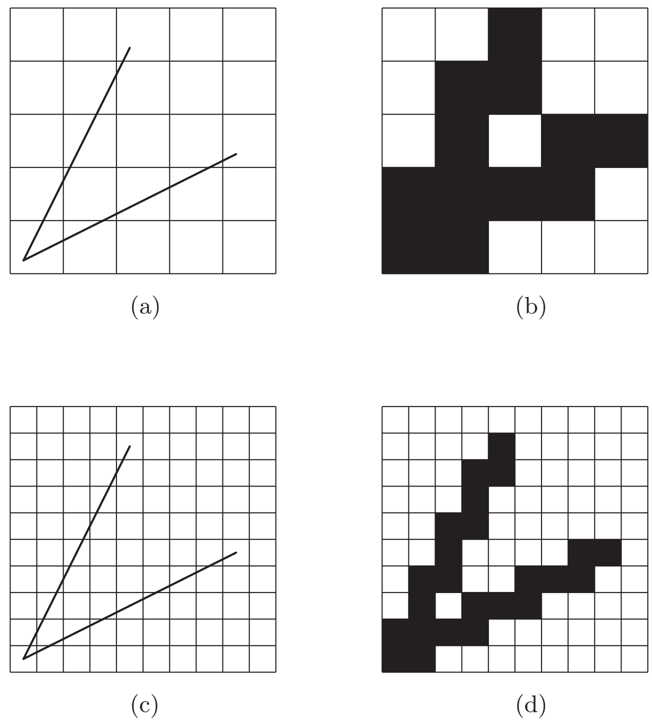


Fig. 8.2. (a) Two slanted line segments that define a set V in a cubical grid. (b) Two dimensional cubes that intersect the set V are colored black. (c) The set $\Lambda^{(2,2)}(V)$ in a cubical grid. (d) Two dimensional cubes that intersect $\Lambda^{(2,2)}(V)$ are colored black.

Example 8.1 Even the simplest process of analog to digital conversion can produce incorrect or spurious information. Since the focus of this book is on computational homology we will show that it is very easy for pixel data to produce the wrong topological information. Consider the two slanted line segments, $V \subset \mathbf{R}^2$, indicated in Figure 8.2(a). Let us think of it as a physical object, for example two line segments drawn in black ink on a piece of white paper. Consider now a highly idealized description of the information in a very coarse digital photo of V. In particular, assume that the digital photo consists of 25 pixels and that each pixel is either black or white depending on whether V intersects the area associated to each pixel. This is indicated in Figure 8.2. How V intersects each pixel is indicated in (a) and the corresponding assignment of black and white is indicated in (b).

Observe that the pixels in Figure 8.2(b) naturally define three sets: $\mathcal{Z}$ which consists of all 25 pixels in the 5×5 region; $\mathcal{X}$ the set of all black pixels; and $\mathcal{Y}$ the set of all white pixels. If we identify each pixel with a two-dimensional elementary cube, then $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ give rise to cubical sets X, Y, and Z defined by (2.10)

$$X := |\mathcal{X}|,\ Y := |\mathcal{Y}|,\ Z := |\mathcal{Z}|.$$

Thus the combinatorial information is transformed to topological information. A word of caution is due here. Notice that $\mathcal{X}$ and $\mathcal{Y}$ are complementary sets of pixels, however the corresponding cubical sets X and Y are not complementary sets of points because they are not disjoint: they have some common vertices and edges. Thus the correspondence between the combinatorial objects and the topological ones is not exact. It might seem that the identification of pixels with two-dimensional open elementary cells could solve the problem of the lack of disjointness. But then $Z = X \cup Y$ would not give the full rectangle: All vertices and edges would be missing. This inconsistency comes from the fact that the information about zero and one-dimensional faces, which is crucial for the computation of homology, cannot be exhibited by pixels themselves because those are the smallest distinguishable units in the numerical picture, often regarded as points.

Let us return to the set V. It is not cubical and therefore we cannot yet compute its homology (in Chapter 11 we will generalize homology to a larger class of spaces). However, it is homotopic to a single point, and thus should have the same homology as a point. On the other hand, we are thinking of X as a cubical approximation or representation of V, but it is clearly not acyclic. Furthermore, we cannot blame the resolution of our camera. Figures 8.2(c) and (d) indicate what happens if we apply the same procedure using a digital photo with 100 pixels. Observe that the homology of the cubical approximation remains unchanged.

The moral of this example is that producing discretized data and then interpreting it in the form of a cubical complex is not a fail safe procedure. On the other hand, as we will explain in the remainder of this chapter, the reader should not assume that this example implies that digital approximations are doomed to failure.

Turning the set V into a cubical complex is perhaps the simplest example of the process of converting an analog object into a digital image. Part of the simplicity comes from the fact that the only characteristic of the letter is its shape. In more general examples, i.e. the black and white photograph of the lunar landscape, one obtains a grey scale image where the pixel values are definitely not binary. To directly apply homology we need to reduce this grey scale image to a binary image.

The simplest approach for reducing a grey scale image such as that of Figure 8.1 to a binary image is that of *image thresholding*. One chooses a range of grey scale values $[T_0, T_1]$ and all pixels whose values lie in $[T_0, T_1]$ are colored black while the others are left white. Of course, the resulting binary image depends greatly on the values of T_0 and T_1 that are chosen. Again, the question of the optimal choice of $[T_0, T_1]$ and the development of more sophisticated reduction techniques is a serious problem that is not dealt with in this book.

Having acknowledged the simplistic approach we are adopting here consider Figure 8.3. This binary images were obtained from Figure 8.1 as follows. Recall that in a grey scale image each pixel has a value between 0 and 255 where 0 is black and 255 is white. We set the threshold intervals to be $[95, 255]$ and $[105, 225]$. Thus, the black pixels correspond to the lightest pixels in the original image and the white pixels to the darkest. As before we will treat the black pixels as defining a cubical complex. It should not come as a surprise that different thresholding intervals result in different cubical complexes.

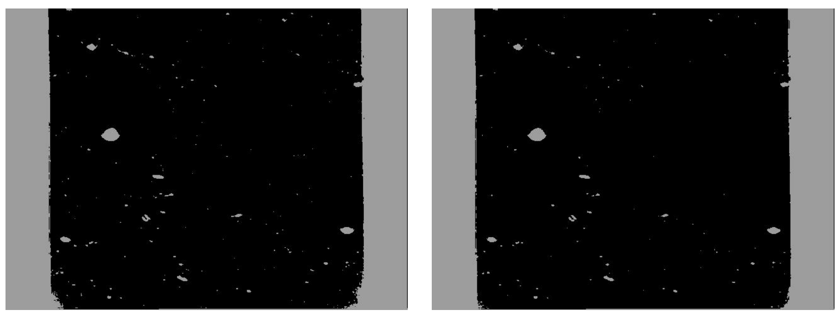


Fig. 8.3. The left figure was obtained from Figure 8.1 by choosing the threshold interval $[95, 255]$ while the right figure corresponds to the threshold interval $[105, 225]$.

Up to this point we have discussed conversion of analogue objects to digital images and then thresholding these images to obtain binary images. We have also emphasized that there are inherent difficulties involved in these reductions. On a more positive side let us consider how the ideas from homology learned in Part I can be used to extract information about the original object from binary images.

Example 8.2 Consider Figure 8.1. One of the more striking features are the craters and a natural question is how many are there. We shall use the binary images in Figure 8.3 to give an approximate answer to this question. Observe that the white pixels correspond to the darker pixels in Figure 8.1. Also observe that the darker regions in Figure 8.1 correspond to shadows interior to the craters.

Let us recast these observations in terms of cubical complexes. Let X be the cubical complex defined by the black pixels of the left image in Figure 8.3. Then, the holes in X correspond to the shadows of the craters. Thus, computing $H_1(X)$ provides us with a count of the number of craters.

This simple example allows us to point out once again the care that needs to be taken in both the imaging and thresholding (we clearly would get a different answer if we used the right image in Figure 8.3). Also, from the point of view of computational efficiency there are better ways than using homology to answer the simple question of how many craters are in the picture. However, even though this example is somewhat artificial we hope the reader has obtained an appreciation for the process of going from an image to homology and back to information about the image. We now turn to more subtle examples involving higher dimensional data where simpler algorithms are not obvious.

In biomedical imaging tomography is a standard tool. While each single tomographic image represents a 2 dimensional slice of the object, a 3 dimensional image can be reconstructed by using a set of parallel contiguous slices. Each single data point is now referred to as a voxel. As before, to a given collection of binary voxel data, we can associate a 3 dimensional cubical complex.

As an example consider the MRI shown in Figure 8.4 This is one out of 124 two-dimensional slices of volumetric image data of the brain. It is easily noted that in contrast to looking at a picture of the moon's surface viewing 3 dimensional images is a non-trivial task. There are of course nice software packages that allow one to view and rotate 3 dimensional objects. Thus, with a sufficiently powerful computer we can easily study the surface of a reasonably complex object. For example, as is indicated in Figure 8.4 a smooth rendering of the brain's surface can be obtained.

However, if we want to be able to simultaneously study the interior of the object, then the problem of visualization becomes much more difficult.

Of course, this is precisely the point of introducing homology groups. Because we can compute them directly from a cubical complex, to determine the cavities of the brain in Figure 8.4 it is sufficient to be given a binary voxel representation of the organ. No visualization is necessary. Using thresholding techniques coonsiderably more sophisticated than those presented earlier a binary image was obtain (a single slice of this binary image is shown in Figure 8.4). The homology of the resulting three dimensional complex was computed, resulting in Betti numbers $\beta_0 = 1$, $\beta_1 = 3$ and $\beta_2 = 1$.

There is another very natural way to obtain 3 dimensional images. Consider a series of two dimensional binary images which are changing with time. If we stack them in chronological order, then we obtain a 3 dimensional image with one of the axis representing time (see Section 8.2).

Finally, if one begins with a 3 dimensional tomographic image that changes with time (think of a beating heart), then the image plus time results in a

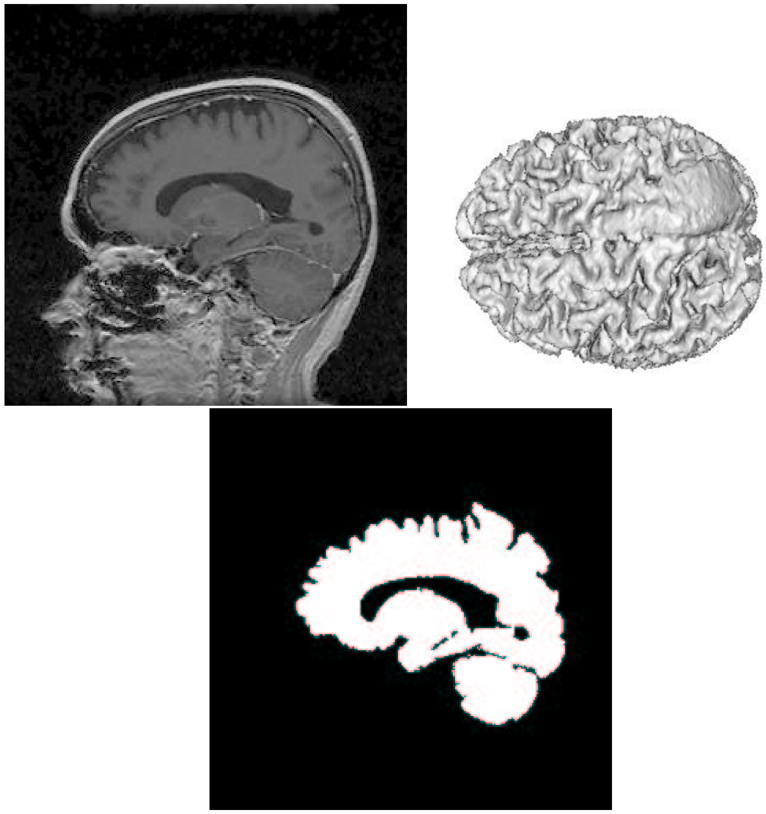


Fig. 8.4. (a) A single two dimesional slice of an MRI image of a brain. The tissue of interest is indicated in light grey. The full data set consists of 256 by 256 by 124 voxels. (b) A view of the surface of the segmented brain. (c) A slice of the segmented binary image on which the homology program can operate. In this case the cubical complex is indicated in white.

4 dimensional object. The individual data points are called *tetrals*. From our point of view such a binary image could be interpreted as a 4 dimensional cubical complex.

8.2 Complicated Time Dependent Patterns

Though we began this chapter with a note of caution, we firmly believe that there are many problems in which homology provides a window by which one can efficiently extract information from complicated problems in which standard imaging techniques are difficult to use. As we suggested earlier, understanding the geometric structure of complicated 3 dimensional objects is difficult. With this in mind we turn our attention to the analysis of complicated time dependent wave patterns.

Spiral waves occur in a wide variety of physical systems including the Belousov-Zhabotinskii chemical reaction, corrosion on metal surfaces, aggregation of slime molds, electrochemical waves in brain tissue, and contractions of muscle cells in cardiac tissue (see [39] especially Chapter 12 and Section 12.7 for an introduction to the biological and mathematical mechanisms behind these phenomena). In the last example, the break up of these waves from fairly simple rotating spirals to highly complicated interactive patterns is associated with the onset of ventricular fibrillation, a potentially fatal cardiac rhythm. Due to their importance an enormous amount of work has been done in an attempt to understand their behavior and much is known. However, as is indicated by the numerically generated images in Figures 8.5 and 8.6 these systems generate very complicated spatial patterns and the structure of these patterns change with time. Thus, obtaining a quantitative description or classification of these waves both spatially and temporally is a very challenging problem.

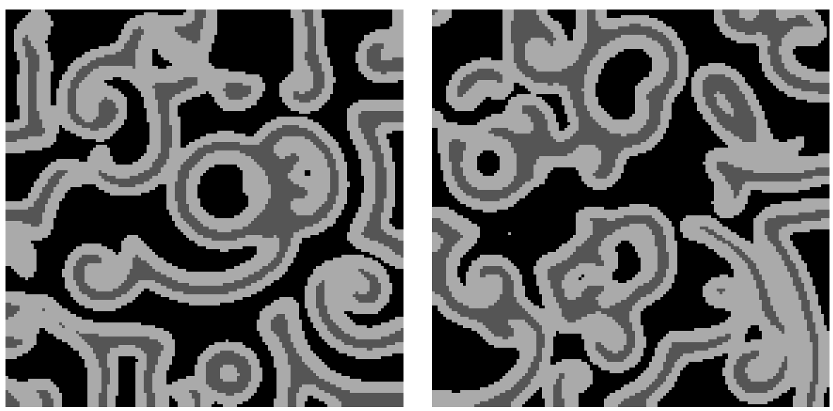


Fig. 8.5. Numerically generated spiral wave with parameter values $a = 0.75$, $b = 0.06$, $\frac{1}{\epsilon} = 12$ and $\nu = 0$. The number of grid points in each direction is 141. The left figure occurs at time t=1000 while the right figure occurs at t=2000. The dark grey represents the excited media, that is $u(x, y) \geq 0.9$.

Figures 8.5 and 8.6 were generated by numerically solving the following system of partial differential equations that provides an extremely simple model for electropotential waves in a 2-dimensional slice of cardiac tissue

$$\frac{\partial u}{\partial t} = \frac{1}{80^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{u}{\epsilon}(1-u)\left(u - \frac{v+b}{a}\right)$$

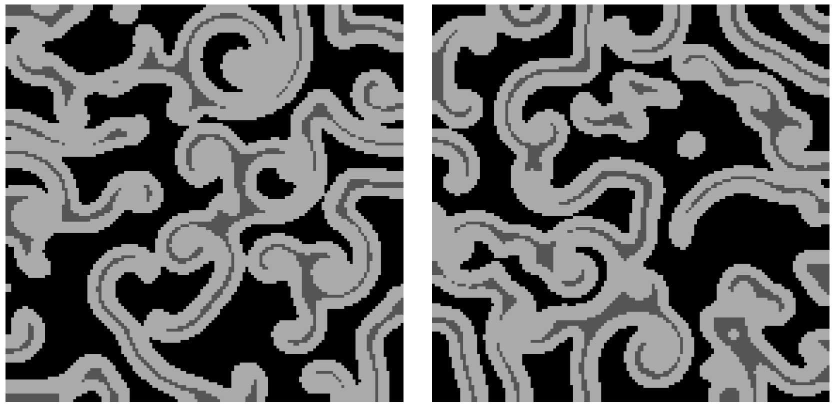


Fig. 8.6. Numerically generated spiral wave with parameter values $a = 0.65$, $b = 0.06$, $\frac{1}{\epsilon} = 12$ and $\nu = 0$. The number of grid points in each direction is 141. The left figure occurs at time t=1000 while the right figure occurs at t=2000. The dark grey represents the excited media, that is $u(x, y) \geq 0.9$.

$$\frac{\partial v}{\partial t} = \frac{\mu}{80^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + u^3 - v$$

where $t \in [0, \infty)$ and $(x, y) \in [0, 1] \times [0, 1] \subset \mathbf{R}^2$ with Neumann boundary conditions, that is

$$\frac{\partial u}{\partial \nu}(x, y, t) = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x, y, t) = 0$$

if (x, y) is on the boundary of $[0, 1] \times [0, 1]$ and ν is the outward normal to the boundary.

We shall not attempt to explain this model here (see [39] for instance) but will only focus on using homology to investigate the complicated behavior exhibited by this system. If the value of the function u exceeds 0.9 at a particular point in time, then we say that the material or media is excited. For a fixed time t the points in the domain $[0, 1] \times [0, 1]$ where $u(x, y, t) \geq 0.9$ are shaded dark grey and we are interested in how these grey regions change with time.

The images obtained in Figures 8.5 and 8.6 were derived from the same initial conditions, but with only the parameter value a changed. Thus, it is obvious that varying just a single parameter produces different wave patterns.

There are two questions that we shall attempt to address. The first is: can we find a characteristic that distinguishes the geometry of the waves at the two parameter values? For example, one could imagine counting the number of components corresponding to the excited media. The second has to do with the temporal variability of the complexity. Obviously, each image exhibits a complex spatial pattern. Is there a way to measure how complicated each pattern is and if so is this level of complexity constant in time or does the level of complexity vary?

To answer these questions the excited media was treated as a 3-dimensional subset of the domain cross time, i.e. as a subset of $[0,1] \times [0,1] \times [0,\infty)$. The justification for this is that in many of the physical examples, for example cardiac tissue, the movement of the waves through the media is of primary importance. The two dimensional images shown in Figures 8.5 and 8.6 cannot capture this information.

Let us be more precise about the procedure that was employed. At each time step the partial differential equation was solved numerically on a grid consisting of 141×141 points. Thus, at time step k there was a collection of values $(u(i,j,k), v(i,j,k)) \in \mathbf{R}^2$, where $i,j \in \{1,\ldots,141\}$, which was updated to a new set of values $(u(i,j,k+1), v(i,j,k+1))$ that represented the discretized solution at the next time step $k+1$. The excited elements of the media were defined to be those points at which $u(i,j,k) \geq 0.9$. Thus the cubical set $X \subset [0,141] \times [0,141] \times [0,\infty)$ was given by

$$[i-1,i] \times [j-1,j] \times [k,k+1] \in \mathcal{K}_3(X) \quad \Leftrightarrow \quad u(i,j,k) \geq 0.9.$$

In this way X represented a cubical approximation of the exited wave.

For each example associated with Figures 8.5 and 8.6 the following procedure was performed. Beginning with the same initial condition the system was numerically integrated for $50,000$ time steps. To better capture the behavior with respect to time, the cubical sets were constructed using the data from time sequences of length 1000. To be more precise, for the first example the cubical sets W_l^1, for $l = 1,2,\ldots,50$, were defined by using the data points $\{u(i,j,k) \mid l \times 1000 \leq k < (l+1) \times 1000\}$ generated when $a = 0.75$. The sets W_l^2 were defined similarly for $a = 0.65$.

Finally, the homology groups $H_*(W_l^i)$ were computed. Since most waves collide with other waves, H_0 provides very little information. Similarly, there are very few volumes actually enclosed by the waves, thus H_2 is not interesting. What is plotted in Figure 8.7 are the first Betti numbers β_1 of W_l^1 and W_l^2 as a function of l, respectively.

With regard to the questions we posed earlier, there are two comments that need to be made. The first is that since the scales on the vertical axis differ by an order of magnitude, it is clear that the wave structures at the two parameter values $a = 0.75$ and $a = 0.65$ are different. The second is perhaps more interesting. Observe that β_1 varies both tremendously and erratically. This suggests that not only does the spatial complexity change with time, but that it changes in a chaotic or unpredictable manner.

8.3 Size Function

In the example of the previous section we gained information by computing the homology groups of a cubical set directly related to a numerical image. We

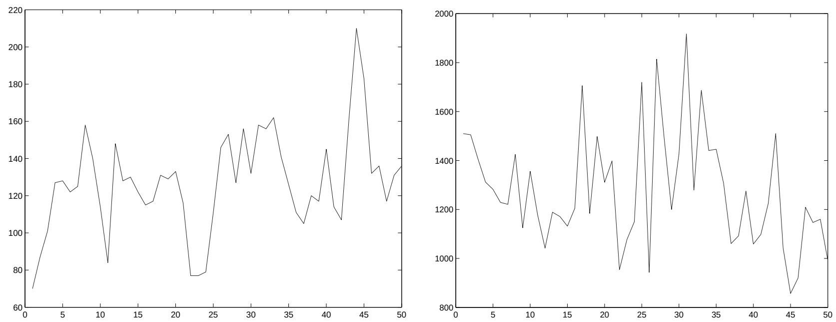


Fig. 8.7. Plots of β_1 of W_l^1 and W_l^2 as a function of l, respectively.

now turn to a problem where the homology group by itself does not provide enough information.

Fig. 8.8. Two sample photographic records of the international sign language alphabet provide courtesy of A. Verri (top) and to C. Uras (bottom). The letter Z is not recorded because it is realized by a dynamical gesture so it requires a video recording rather than a single photography.

Consider two sample photographic records of the *International Sign Language* alphabet presented in Figure 8.8. There is an extensive body of research devoted to developing an automated approach to recognizing hand gestures from images as meaningful signs. A common approach to pattern recognition is based on matching an image with a known model. Unfortunately these methods have a high failure rate whenever the relation between the observed object and the model cannot be defined precisely. In the sign language, the shapes of hands showing the same sign may vary considerably depending on the hand of a particular person, the precision of the gesture, and the light or angle under which the photograph has been taken. Therefore, for this problem the tolerance in matching objects needs to be quite high. On the other hand, it cannot be too high because many signs are very similar in shape. Compare for example the signs "K" and "V". In both, two fingers are pointed straight, though at different angles. A tool to compare these signs must be insensitive to small variations of angles under which fingers are pointed out, but it cannot be completely rotation-invariant, otherwise we can not distinguish between the two signs.

Topology might seem to be a proper tool for comparing given shapes with model shapes due to its insensitivity to size. In the case of a sign langauge, we can not advance much by directly applying homology to images of the hands in Figure 8.8 because all except for the letter "O" represent contractible sets. Hence their homology groups are all trivial. We shall present here a more subtle approach based on the concept of *size functions* introduced by Frossini [19] (see also [21] and the references therein) and applied to sign language by Uras and Verri [45].

We start from the following preliminary observation. Due to the imperfection of photography and variable levels of light, the information given by shades of gray in the image of a hand can be very misleading. The most reliable information provided by the techniques of numerical imaging is the information on the contour limiting a hand image. With this in mind we change our focus from the full images of the hands shown in Figure 8.8 to their contours as indicated in Figure 8.9.

The curves in the latter figure are still fairly complicated and so for the sake of simplicity, we shall describe the basic concepts of sign functions using the two contours in $\mathbf{R}^2$ indicated in Figure 8.10.

We start by a continuous function $f : \mathbf{R}^2 \to [0, \infty)$ which is called a *measuring function*. In this example our choice is the function $f : \mathbf{R}^2 \to [0, \infty)$,

$$f(x) = \|x - a\|_0. \tag{8.1}$$

measuring the distance from a given *reference point* $a \in \mathbf{R}^2$ to x. In Figure 8.10, the reference points a_0 and a_1 are chosen to be within the contours E_0 and E_1 respectively.

We postpone the discussion on the pertinence of this particular choice of measuring function and limit ourselves to the remark that the difference in shape of E_0 and E_1 can be observed in the fact that the values of this function

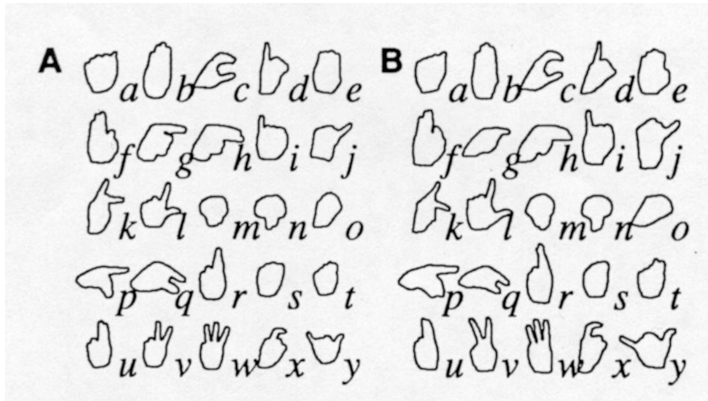


Fig. 8.9. The hand contours corresponding to two recordings of the sign language alphabet on Figure 8.8.

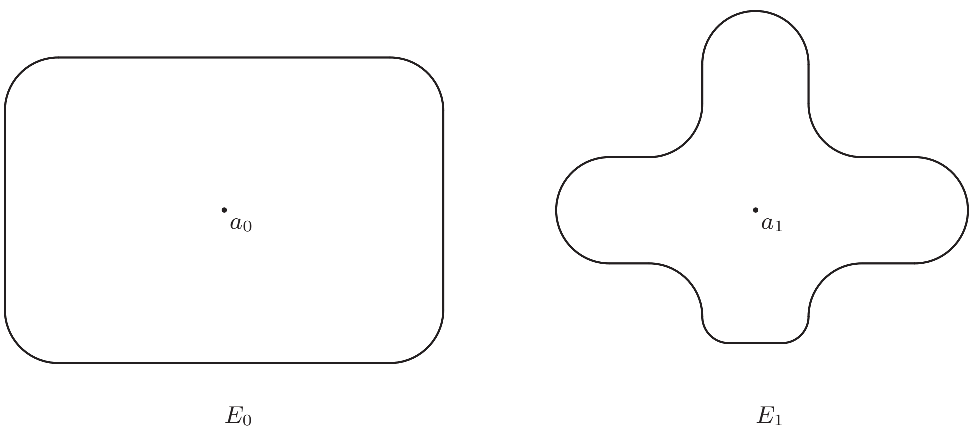


Fig. 8.10. Two distinct contours E_0 and E_1 along with reference points a_0 and a_1.

have greater oscillation on E_1 than on E_0. With this in mind we now present a method for measuring these levels of oscillation using topology.

Consider a bounded subset $E \subset \mathbf{R}^2$. For $\alpha \in [0, \infty)$, define *level sets* of f in E by

$$E^\alpha = \{x \in E \mid 0 \leq f(x) \leq \alpha\}.$$

We will study how the topology of the level sets change as the value of the parameter α changes. Since we will only apply the theory to a contour E in

$\mathbf{R}^2$, the level sets will be pieces of that contour. We will concentrate on the number of connected components of the level sets.

For a fixed $\beta \geq \alpha$ define an equivalence relation in E^α as follows,

$$x \overset{\beta}{\sim} y \text{ if } x \text{ and } y \text{ belong to the same connected component of } E^\beta.$$

We will study this relation for the pairs (α, β) where $\beta \geq \alpha \geq 0$ are either reals or integers.

Definition 8.3 The function $n : \{(\alpha, \beta) \in \mathbf{R}^2 \mid \beta \geq \alpha \geq 0\} \to \mathbf{Z}$ given by

$$n(\alpha, \beta) = \text{number of equivalence classes of } E^\alpha \text{ under } \overset{\beta}{\sim}.$$

n is the *size function* associated to the set E and the measuring function f.

Since we want to be able to compute the values of the sign function, we must somewhat discretize studied contours. The approach we have chosen in this chapter is approximating a given set by a cubical set. Given a bounded set $E \subset \mathbf{R}^2$ let

$$\tilde{E} := \operatorname{ch}(E).$$

Recall that this is the smallest cubical set containing E.

Since we restrict the study of size function to cubical sets, we will also restrict the values of α and β to nonnegative integers $\mathbf{Z}^+$. This assumption is reasonable if we take the point of view that the features we wish to extract are large with respect to the grid size. Restating this as an imaging problem, the assumption is that the objects of interest are represented by a significant number of pixels. Thus we shall study the equivalence relation $x \overset{\beta}{\sim} y$ on the sets $\tilde{E}^\alpha$, where we assume now that α and β are in $\mathbf{Z}^+$. We may also assume that the reference point a for our measuring function given in (8.1) has integer coordinates. It is left as an exercise to show that $\tilde{E}^\alpha$ is a cubical set for any integer α. Finally, let

$$\tilde{n} : \{(\alpha, \beta) \in \mathbf{Z}^2 \mid \beta \geq \alpha \geq 0\} \to \mathbf{Z}$$

be the restriction of the size function n associated with $\tilde{E}^\alpha$ and d to integer values of α and β.

In Figure 8.11 we have placed the curves E_0 and E_1 along with the points $a_0 = (0, 0)$ and $a_1 = (0, 0)$ into cubical grids and used yellow to indicate the sets $\tilde{E}_0$ and $\tilde{E}_1$.

To develop some intuition as to what the size function measures, set $\alpha = 4$ and $\beta = 7$. In Figure 8.12, the sets $\tilde{E}_i^\alpha$ are indicated in red and the sets $\tilde{E}_i^\beta \setminus \tilde{E}_i^\alpha$ are indicated in blue. Observe that $\tilde{E}_i^\beta$ consists of those points in the original curve that lie in the blue or red squares. Notice that there are no red squares in Figure 8.12(a) and thus $\tilde{E}_0^4 = \emptyset$. Therefore,

$$\tilde{n}_0(4, 7) = 0.$$

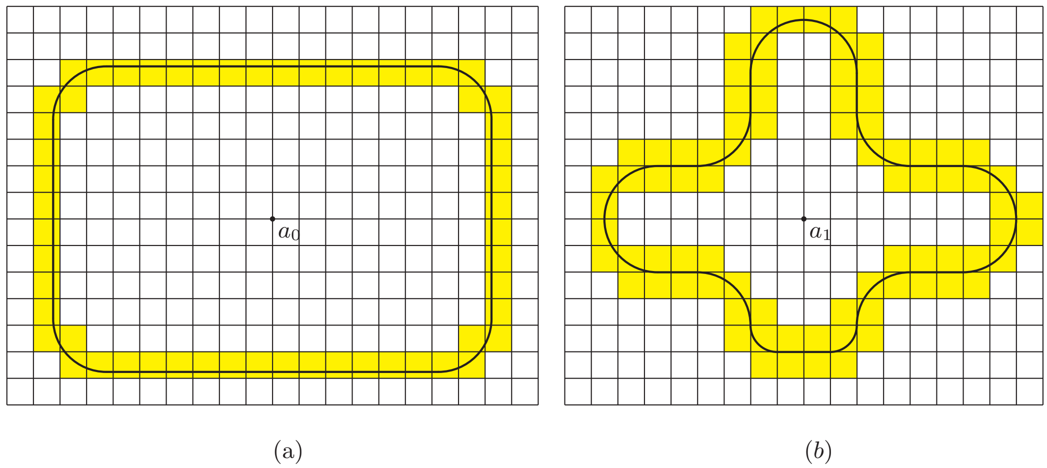


Fig. 8.11. The two distinct closed curves E_0 and E_1 placed in cubical grids along with reference points a_0 and a_1. For $i = 0, 1$, the cubical sets $\tilde{E}_i$ are indicated in yellow.

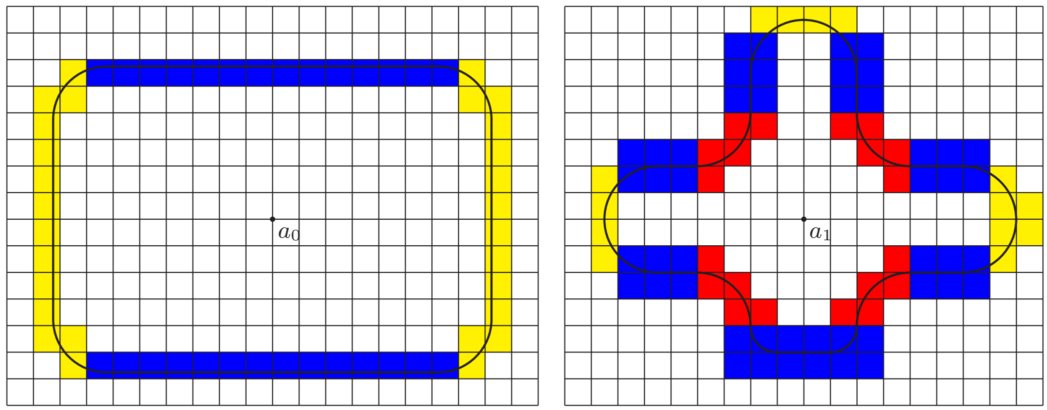
On the other hand, as Figure 8.12(b) indicates $\tilde{E}_1^4$ consists of 4 components, but two of the four components are connected to the same blue component, thus

$$\tilde{n}_1(4, 7) = 3.$$

Observe, that having chosen the grid and measuring function, we can make tables of values as is indicated in Figure 8.13 of $\tilde{n}_i(\alpha, \beta)$ for $\alpha, \beta \in \mathbf{Z}^+$. The second table shows more variation of the values of our size function. This can be related to the fact that $\tilde{E}_1$ wiggles more away and towards the center than $\tilde{E}_0$.

Thus the size function may provide an automatic way of distinguishing curves of different shapes. Does it necessarily do so? In our example it did but, in general, one particular choice of a measuring function might not be sufficient to distinguish properties we care about. Consider for example an image of a handwritten letter **p**. If our measuring function is invariant under rotation about the x-axis, then the corresponding size function would not permit us to distinguish it from an image of a handwritten letter **b**. Thus in practice a collection of measuring functions is applied to compared images and the data coming from the corresponding size functions is studied by techniques of statistics.

For example, in [45] a measuring function f_0 was defined as the euclidean distance from a point x to the lower half-space bounded by the horizontal line L passing through a reference point a. Then a family of 72 measuring functions



(a) (b)

Fig. 8.12. The colored squares indicate the cubical approximations $\tilde{E}_0$ and $\tilde{E}_1$ of the two simple curves E_0 and E_1. The reference points are $a_0 = (0,0)$ and $a_1 = (0,0)$. The red cubes indicate $\tilde{E}_i^4$. The blue cubes indicate $\tilde{E}_i^7 \setminus \tilde{E}_i^4$.

$\beta \setminus \alpha$	6	7	8	9
6	2			
7	2	2		
8	2	2	2	
9	1	1	1	1

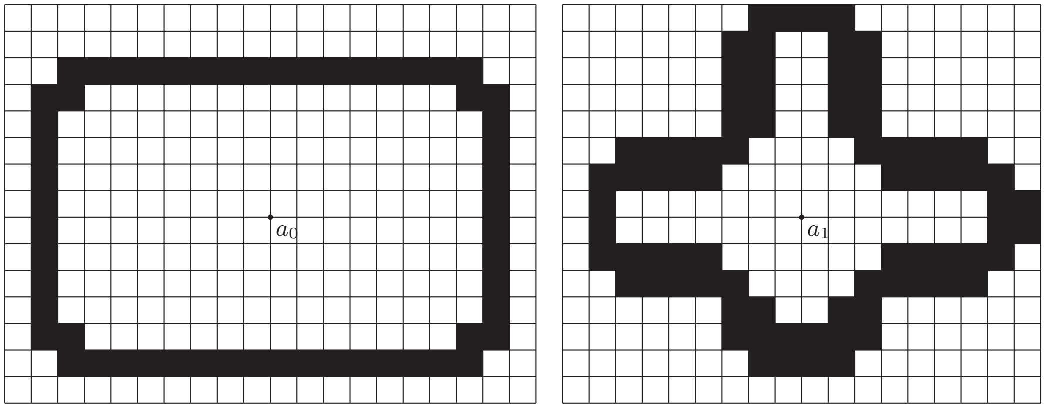
$\beta \setminus \alpha$	3	4	5	6	7	8	9
3	4						
4	4	4					
5	3	3	3				
6	3	3	3	3			
7	3	3	3	3	3		
8	1	1	1	1	1	1	
9	1	1	1	1	1	1	1

Fig. 8.13. Size function tables for $\tilde{E}_0$ (left) and $\tilde{E}_1$ (right). The values of α and β for which $\tilde{n}_i = 0$ have not been included.

f_θ was defined by rotating the line L around the reference point by angles $\theta = 0°, 5°, 10°, \ldots, 355°$ and statistical analysis of results was performed.

Our discussion of the size function began with a contour E. This was done in an attempt to provide the reader with an intuitive understanding of this tool. However, from the point of view of image processing it is more realistic to think of the starting point as a digitalized image. With all the caveats of Example 8.1 still in mind we shall go one more step and assume that we begin with binary images such as those indicated in Figure 8.14. Thus, our goal is to distinguish or identify the associated cubical sets using size functions.

By now the reader may be asking where does the homology come into play? Recall that we know how to count connected components using homology. In



(a) (b)

Fig. 8.14. The two images presented as cubical sets $\tilde{E}_0$ and $\tilde{E}_1$ along with reference points a_0 and a_1.

particular, by Theorem 2.59 the number of connected components equals the rank of $H_0(X)$. The following notation will prove useful. Given a pair of cubical sets $A \subset X$ let $q(X, A)$ be the number of connected components of X which are disjoint from A.

Proposition 8.4

$$\tilde{n}(\alpha, \beta) = \operatorname{rank} H_0(\tilde{E}^\beta) - q(\tilde{E}^\beta, \tilde{E}^\alpha)$$

Proof. Since we assume $\beta \geq \alpha$, it follows that $\tilde{E}^\alpha \subset \tilde{E}^\beta$. Thus, any connected component of $\tilde{E}^\alpha$, being a connected set, must be contained in a connected component of $\tilde{E}^\beta$, Hence, it is contained in the same equivalence class of the relation $x \overset{\beta}{\sim} y$. Of course, if two connected components of $\tilde{E}^\alpha$ lie in the same connected component of $\tilde{E}^\beta$, then they give rise to the same equivalence class. Thus, $\tilde{n}$ is the number of those connected components of $\tilde{E}^\beta$ that contain at least one connected component of $\tilde{E}^\alpha$. Obviously $\operatorname{rank} H_0(\tilde{E}^\beta)$ is the sum of the number of connected components of $\tilde{E}^\beta$ that contain a component of $\tilde{E}^\alpha$ and the number of connected components of $\tilde{E}^\beta$ that do not contain any. But a connected component B of $\tilde{E}^\beta$ which does not contain any connected component of $\tilde{E}^\alpha$ must be disjoint from $\tilde{E}^\alpha$. Indeed, suppose that B is not disjoint from $\tilde{E}^\alpha$ and let $a \in B \cap \tilde{E}^\alpha$. Let A be the connected component of $\tilde{E}^\alpha$ containing a. Then $A \cup B$ is a connected set containing B (see Exercise 12.25). By the maximality, $A \cup B = B$ so $A \subset B$, so B contains a connected component of $\tilde{E}^\alpha$. Hence

$$\operatorname{rank} H_0(\tilde{E}^\beta) = \tilde{n}(\alpha, \beta) + q(\tilde{E}^\beta, \tilde{E}^\alpha)$$

and the conclusion follows. □

In the light of Proposition 8.4, in order to provide a purely homological formula for computing $\tilde{n}$ we need a homological method for computing $q(X, A)$ where $A \subset X$. This will be provided for in Chapter 9 where the concept of relative homology is introduced.

We finish this section with some remarks that are intended to put our discussion of size functions into proper perspective. As presented here size functions are related to one topological feature - the variation in the number of connected components of level sets of measuring functions. For this purpose 0-dimensional homology is sufficient. Of course, from the point of view of computations there are more efficient algorithms for counting connected components than directly computing H_0. On the other hand, when one considers higher dimensional images a wider menagerie of topological features and higher dimensional homologies come into the play. Generalizing size functions in this direction leads to Morse theory.[1] Morse theory is an extremely rich topic and one of the fundamental tools of differential topology. The reader interested in pursuing this topic from a mathematical point of view is encouraged to consult [34, 7, **?**]. On the computational side there are no standard references. The subject is too new and is still undergoing rapid advances both theoretically and in terms of it domain of application. However the reader may find the following articles of interest [2, 16].

Exercises

8.1 Let X be a cubical set in $\mathbf{R}^2$. Consider the following measuring functions on X.

(a) $f(x) := x_2$;
(b) $f(x) := ||x||_0$.

Show that the associated sets X^α are cubical sets for any nonnegative integer α.

8.2 * Let X be a cubical set in $\mathbf{R}^2$ and let f be the size function considered in Exercise 8.1(a). Prove that the associated size function only depends on the integer parts $\operatorname{floor}\alpha$ and $\operatorname{floor}\beta$ of, respectively, α and β.

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[1] This link is discussed explicitly in [20].

9

Homological Algebra

We finished the previous chapter with the observation that given a space X and a subset A it would be useful to have a means of computing the number of components of X that are disjoint from A. Using this as motivation, in Section 9.1 we introduce the concept of relative homology. Though motivated by an a simple problem, it turns out that relative homology is an extremely powerful tool, both as a means of extracting topology from the homological algebra and as an abstract computational technique. The language of these computational methods takes the form of exact sequences which are discussed in Section 9.2.

The reader may recall that by Theorem 2.78 if X, Y and $X \cap Y$ are acyclic sets, then $X \cup Y$ is acyclic. This allows one to compute the homology of a large set in terms of subsets, but at the price of acyclicity. The Mayer-Vietoris sequence presented in Section 9.4 is the generalization of this theorem - it allows one to compute the homology of $X \cup Y$ in terms of X, Y and $X \cap Y$ for arbitrary cubical sets. However, to obtain this sequence requires a fundamental algebraic result, the construction of the connecting homorphism, described in Section 9.3.

9.1 Relative Homology

Given a pair of cubical sets $A \subset X$ we will define the relative homology groups $H_*(X, A)$. These groups can be used directly to measure how the topological structure of X and A differ. In Subsection 9.1.2 we will consider the case of another pair of cubical sets $D \subset Y$ along with a continuous map $f : X \to Y$ with the property that $f(A) \subset Y$ and show that there is an induced homomorphism on the relative homology groups.

9.1.1 Relative Homology Groups

Let us return to the task set forth in the previous section of using ideas from homology to count the number of components in X that do not intersect a subset A. Since we will often want to talk about a pair of cubical sets one of which is the subset of the other the following definition is useful.

Definition 9.1 A pair of cubical sets X and A with the property that $A \subset X$ is called a *cubical pair* and denoted by (X, A).

The discussion that follows is easier to understand if we have a specific example to work with. So consider the cubical pair (X, A) in Figure 9.1. Both A and X have four components. Since the only component of X that does not intersect a component of A is X_3, we would like to end up with the number 1.

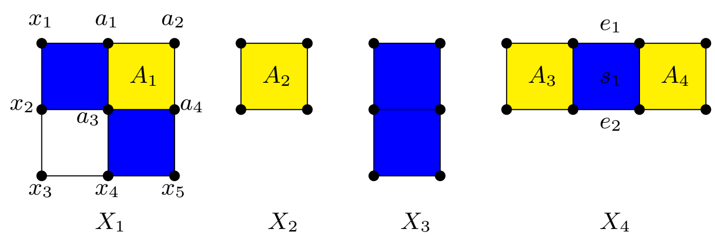

Fig. 9.1. Cubical sets $A \subset X$. A is indicated in yellow. Observe that both A and X have four components. The only component of X that does not intersect a component of A is X_3.

Let us begin by restricting our attention to the component X_2. Referring to Figure 9.1 it is clear that $X_2 = A_2$ and thus X_2 should not be counted. Of course this needs to be done on the algebraic level. Thus, we start with the cubical chains $C_*(X_2)$ and make the observation that since $C_*(A_2) = C_*(X_2)$ we want to trivialize $C_*(X_2)$. The most natural algebraic method for doing this is to consider the quotient group $C_*(X_2)/C_*(A_2) = 0$. Obviously, on the algebraic level the component X_2 has disappeared.

Observe that we can generalize this simple operation. Associated to the cubical pair (X, A) are sets of elementary cubes $\mathcal{K}(X)$ and $\mathcal{K}(A)$ which in turn define the chains $\mathcal{C}(X)$ and $\mathcal{C}(A)$. Since $\widehat{\mathcal{K}}_k(X)$ is a basis for $\mathcal{C}(X)$ and $\widehat{\mathcal{K}}_k(A) \subset \widehat{\mathcal{K}}(X)$, the quotient group $\mathcal{C}(X)/\mathcal{C}(A)$ is a free abelian group (see Exercise 13.17). Thus we can make the following definitions.

Definition 9.2 Let (X, A) be a cubical pair. The *relative chains of X modulo A* are the elements of the free abelian groups

$$C_k(X, A) := C_k(X)/C_k(A).$$

The equivalence class of a chain $c \in \mathcal{C}(X)$ relative to $\mathcal{C}(A)$ is denoted by $[c]_A$.

The *relative chain complex of X modulo A* is given by

$$\{C_k(X,A), \partial_k^{(X,A)}\}$$

where $\partial_k^{(X,A)} : C_k(X,A) \to C_{k-1}(X,A)$ is the boundary map induced by the standard boundary map on $C_k(X)$. It is left to the reader to verify that the induced boundary map is well defined and

$$\partial_{k-1}^{(X,A)} \partial_k^{(X,A)} = 0.$$

The relative chain complex gives rise to the *relative k-cycles*,

$$Z_k(X,A) := \ker \partial_k^{(X,A)},$$

the *relative k-boundaries*,

$$B_k(X,A) := \operatorname{im} \partial_{k+1}^{(X,A)},$$

and finally the *relative homology groups*

$$H_k(X,A) := Z_k(X,A)/B_k(X,A).$$

Because $\partial^{(X,A)}$ is induced by the canonical boundary operator ∂, it is common to simplify the notation and let $\partial = \partial^{(X,A)}$. We, too, will adopt this convention from here on, unless it is important to make the distinction.

This is not the first time that we make use of a construction of quotient groups. What is of interest at this point what topological information has been gained by using relative homology. Let us consider our original question - for the cubical pair (X,A) can we count the number of components of X which do not intersect A? From Section 2.3 we know that the dimension of $H_0(X)$ equals the number of connected components of X. Thus we will focus on $H_0(X,A)$. Returning to the example in Figure 9.1 observe that

$$C_0(X) = \bigoplus_{i=1}^{4} C_0(X_i) \quad \text{and} \quad C_0(A) = \bigoplus_{i=1}^{4} C_0(A_i)$$

Furthermore,

$$C_0(X,A) = C_0(X_1)/C_0(A_1) \oplus C_0(X_2)/C_0(A_2) \oplus C_0(X_3) \oplus C_0(X_4)/\left(C_0(A_3) \oplus C_0(A_4)\right)$$

Passing to homology

$$H_0(X,A) = H_0(X_1,A_1) \oplus H_0(X_2,A_2) \oplus H_0(X_3) \oplus H_0(X_4, A_3 \cup A_4).$$

Since $C_0(X_2)/C_0(A_2) = 0$, $H_0(X_2,A_2) = 0$. Referring again to Figure 9.1 $\{\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_5}, \widehat{a_1}, \ldots, \widehat{a_4}\}$ is an obvious set of generators of $C_0(X_1)$, while $\{\widehat{a_1}, \ldots, \widehat{a_4}\}$ generates $C_0(A_1)$. Thus, $C_0(X_1)/C_0(A_1)$ is generated by $\{\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_5}\}$ and furthermore, $[\widehat{a_i}]_{A_1} = 0$. This means that $[\widehat{a_i}]_{A_1}$ will generate $0 \in H_0(X_1, A_1)$. Given a_i and x_j there exists a path of edges from a_i to x_j. Hence, $[\widehat{x_j}]_{A_1}$ generate the same homology class as $[\widehat{a_i}]_{A_1}$. Therefore, $H_0(X_1, A_1) = 0$. A similar argument applies to $H_0(X_4, A_3 \cup A_4)$. Thus,

$$H_0(X, A) \cong \mathbf{Z}.$$

In particular, the dimension of $H_0(X, A) = 1$. Thus for this particular example we can count components using relative homology. The following results show that this is true in general.

Proposition 9.3 *Let X be a connected cubical set and let A be a non-empty cubical subset of X. Then,*

$$H_0(X, A) = 0.$$

Proof. To compute $H_0(X, A)$ we begin by examining the associated set of cycles $Z_0(X, A)$. Since $\partial_0 = 0$,

$$Z_0(X, A) = C_0(X, A) = C_0(X)/C_0(A).$$

From the proof of Theorem 2.59, X connected implies that for any pair $\widehat{P}, \widehat{Q} \in C_0(X)$, there exists $c \in C_1(X)$ such that

$$\partial c = \widehat{P} - \widehat{Q}. \tag{9.1}$$

Since $A \neq \emptyset$, there exists $Q \in \mathcal{K}_0(A)$.

By definition, $\left[\widehat{Q}\right]_A = 0 \in Z_0(X, A)$. Therefore, by (9.1), given any $P \in \mathcal{K}_0(X)$, the homology class of $[\widehat{P}]_A$ is

$$\left[[\widehat{P}]_A\right] = \left[[\widehat{Q}]_A\right] = 0 \in H_0(X, A)$$

Therefore, $H_0(X, A) = 0$. □

Proposition 9.4 *Let (X, A) be a cubical pair. Then the number of connected components of X which do not intersect A is the dimension of $H_0(X, A)$.*

Proof. Let $X = \cup_{i=1}^n X_i$ where X_i are disjoint connected components. Let $A_i = X_i \cap A$. Then

$$H_0(X, A) = \bigoplus_{i=1}^n H_0(X_i, A_i).$$

There result follows from Proposition 9.3. □

As the following theorem indicates relative homology also provides us with a new insight into the definition of reduced homology $\tilde{H}(X)$ studied in Section 2.7.

Theorem 9.5 *Let X be a cubical set and $P \in \mathcal{K}_0(X)$ a chosen vertex of X. Then*

$$H(X,P) \cong \tilde{H}(X).$$

More precisely, we have the identities

$$\tilde{H}_k(X) = H_k(X) = H_k(X,P) \text{ for all } k \geq 1. \tag{9.2}$$

and the isomorphism

$$H_0(X,P) \cong \tilde{H}_0(X). \tag{9.3}$$

induced by the bijection

$$\{[\widehat{Q}]_P \mid Q \in \mathcal{K}_0(X) \setminus \{P\}\} \;\leftrightarrow\; \{\widehat{Q} - \widehat{P} \mid Q \in \mathcal{K}_0(X) \setminus \{P\}\} \tag{9.4}$$

of bases for relative 0-cycles and, respectively, reduced 0-cycles.

Proof. Since $C_k(P) = 0$ for all $k \neq 0$, we have

$$C_k(X,P) = C_k(X) \text{ for all } k \geq 1,$$

and

$$\partial_k^{(X,P)} = \partial_k \text{ for all } k \geq 2,$$

By Definition 2.94 of the reduced cubical chain complex we get

$$\tilde{C}_k(X) = C_k(X) = C_k(X,P) \text{ for all } k \geq 1$$

and

$$\tilde{\partial}_k = \partial_k = \partial_k^{(X,P)} \text{ for all } k \geq 2.$$

hence the identities in 9.2 follow for all $k \geq 2$. The identity for $k = 1$ is a bit more delicate. We know that $\tilde{H}_1(X) = H_1(X)$ but the previous argument only shows that $B_1(X,P) = B_1(X)$. So we need to show that $Z_1(X,P) = Z_1(X)$. For, let $c \in C_1(X) = C_1(X,P)$. The condition $\partial_1^{(X,P)}(c) = 0$ is equivalent to $\partial_1(c) \in C_0(P)$. But $C_0(P) = \mathbf{Z}\widehat{P}$. Therefore $\partial_1^{(X,P)}(c) = 0$ if and only if there exists $n \in \mathbf{Z}$ such that $\partial_1(c) = n\widehat{P}$. If this is the case, then the definition of $\tilde{\partial}$ implies that

$$0 = \tilde{\partial}_0(\tilde{\partial}_1(c)) = \epsilon(n\widehat{P}) = n.$$

Hence $\partial_1^{(X,P)}(c) = 0$ if and only if $\partial_1(c) = 0$.

Finally we show 9.3. by Exercise 13.17 the set on the right-hand side of (9.4) is a basis for $Z_0(X,P)$. We leave as exercise the verification that the set on the left-hand side of (9.4) is a basis for $\tilde{Z}_0(X) := \ker \epsilon$. The formula

$$\phi(\widehat{Q} - \widehat{P}) := [Q]_P$$

defines a bijection between the two bases and so it extends by linearity to the isomorphism $\phi : \tilde{Z}_0(X) \to Z_0(X,P)$. It remains to show that $\phi(\tilde{B}_0(X)) =$

$B_0(X,P)$. Indeed, any boundary in $\tilde{B}_0(X) = B_0(X)$ is a linear combination of boundaries of dual edges, and those are differences of dual vertices. So consider an edge E with $\partial_1(\widehat{E}) = \widehat{Q_1} - \widehat{Q_0}$. Then

$$\begin{aligned}\phi(\partial_1(\widehat{E})) &= \phi\left((\widehat{Q_1} - \widehat{P}\right) - (\widehat{Q_0} - \widehat{P})) = [\widehat{Q_1}]_P - [\widehat{Q_0}]_P \\ &= [\partial_1 \widehat{E}]_P = \partial_1^{(X,P)}[\widehat{E}]_P = \partial_1^{(X,P)}\widehat{E}.\end{aligned}$$

□

Given a cubical pair (X,A) Proposition 9.4 provides us with a complete topological interpretation of the algebraic quantity $H_0(X,A)$. We would also like to have some intuition concerning $H_k(X,A)$ for $k \geq 1$ and A different from a point. With this in mind we consider the following two examples, the generalization of which is left as an exercise.

Example 9.6 Let $X = [0,1]$ and $A = \{0,1\}$. The set of elementary chains $\left\{\widehat{0}, \widehat{1}\right\}$ is a basis for both $C_0(X)$ and $C_0(A)$. Therefore, $C_0(X,A) = 0$. Clearly, $\widehat{[0,1]}$ is a basis for $C_1(X)$ while $C_1(A) = 0$. Therefore, $\left[\widehat{[0,1]}\right]_A$ is a basis for $C_1(X,A)$. Since $C_0(X,A) = 0$, $\left[\widehat{[0,1]}\right]_A$ is a basis for $Z_1(X,A)$ and hence $H_1([0,1],\{0,1\}) = \mathbf{Z}$. Thus,

$$H_k([0,1],\{0,1\}) = \begin{cases} \mathbf{Z} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 9.7 Let $X = [0,1]^2$ and $A = \operatorname{bd}([0,1]^2)$. The reader can check that $C_k(X) = C_k(A)$ for all $k \neq 2$. Thus, $C_k(X,A) = 0$ for all $k \neq 2$. On the other hand $\widehat{[0,1]^2}$ is a basis for $C_2(X)$ while $C_2(A) = 0$. Therefore, $\left[\widehat{[0,1]^2}\right]_A$ is a basis for $C_2(X,A)$. Since $C_1(X,A) = 0$, $\left[\widehat{[0,1]^2}\right]_A$ is a basis for $Z_2(X,A)$ and hence $H_2([0,1],\{0,1\}) = \mathbf{Z}$. Thus,

$$H_k([0,1],\{0,1\}) = \begin{cases} \mathbf{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

From these two examples we see that relative homology can be used to capture important topological information about otherwise acyclic sets. In each case, by an appropriate choice of A, the non-triviality of $H_*(X,A)$ indicated the dimension of $\mathcal{K}_{\max}(X)$.

Example 9.8 Consider the cubical pair (Y,D) where $Y = [-3,3] \subset \mathbf{R}$ and $D = [-3,-1] \cup [2,3]$. Let us compute $H_*(Y,D)$. Since $\mathcal{C}_k(Y) = 0$ for $k \geq 2$, $H_k(Y,D) = 0$ for $k \geq 2$. By Proposition 9.3, $H_0(Y,D) = 0$. Thus, only $H_1(Y,D)$ remains to be determined. However, since $C_2(Y,D) = 0$, $H_1(Y,D) = Z_1(Y,D)$.

Let $z := \widehat{[-1,0]} + \widehat{[0,1]} + \widehat{[1,2]}$ and $\zeta := [z]_D$. It is easy to verify that $\partial c = \widehat{[2]} - \widehat{[-1]}$ which is in $C_0(D)$ so $\partial^{(Y,D)}\zeta = 0$. Thus $\zeta \in Z_1(Y,D)$. We want to show that ζ generates $Z_1(Y,D)$ which would permit us to conclude that $H_1(Y,D) = Z_1(Y,D) \cong \mathbf{Z}$. For, let $c \in C_1(Y)$ be a chain with $[c]_D = 0$. One can decompose c as

$$c = c_1 + c_2 + c_3$$

where c_1 is supported in $[-3,-1]$, c_2 in $[-1,2]$ and c_3 in $[2,3]$. Then $[c_1]_D = [c_3]_D = 0$ so their boundaries are also zero. Consequently, $\partial^{(Y,D)}[c]_D = \partial^{(Y,D)}[c_2]_D = 0$. Next,

$$c_2 = \alpha_1 \widehat{[-1,0]} + \alpha_2 \widehat{[0,1]} + \alpha_3 \widehat{[1,2]}.$$

The condition $\partial c_2 \in C_0(D)$ implies that $\alpha_1 = \alpha_2 = \alpha_3$ so forcingly $c_2 = \alpha_1 z$. Finally, $[c]_D = [c_2]_D = \alpha_1 \zeta$.

Let $X = [-1,2]$ and $A = \{-1,2\}$. Performing a computation similar to that of the previous example leads to the following observation.

$$H_*([-3,3],[-3,-1] \cup [2,3]) \approx H_*([-1,2],\{-1,2\})$$

One can ask if it is merely a coincidence that these homology groups are isomorphic or whether there is a deeper underlying relationship. Since the relative chains of the pair (X,A) are formed by quotienting out by those elementary chains which lie in the subspace A, it seems reasonable to conjecture that if one adds the same cubes to both X and A, then the group of relative chains does not change and hence the homology should not change. Theorem 9.12, presented shortly confirms this, though at first glance its statement may appear somewhat different.

Finally, the reader may wonder why we did not continue with the example of Figure 9.1. It can be shown that

$$H_k(X,A) = \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ \mathbf{Z}^2 & \text{if } k = 1, \\ 0 & \text{otherwise} \end{cases}$$

and, in fact, the reader can perform the necessary computations to check this. However, consider the complexity of the computations of Example 9.8 and then consider the complexity of carrying out the computations for the complex from Figure 9.1 which has numerous 2 dimensional cubes. What would be nice is to have a technique that would allow us to compute $H_*(X,A)$ using our knowledge of $H_*(X)$ and $H_*(A)$. This will be done in Corollary 9.24.

9.1.2 Maps in Relative Homology

Of course, to compare the relative homology groups of different cubical pairs we need to be able to talk about maps. So let (X,A) and (Y,D) be cubical pairs, and let $f : X \to Y$ be a continuous map. The most basic question is whether f induces a map from $H_*(X,A)$ to $H_*(Y,D)$. In order to answer that question, we should first look at a chain map $\varphi : \mathcal{C}(X) \to \mathcal{C}(Y)$ and find out when does it induce a chain map $\bar{\varphi} : \mathcal{C}(X,A) \to \mathcal{C}(Y,D)$. The relative chain groups are quotient groups so the following holds true.

Proposition 9.9 *Let (X, A) and (Y, D) be cubical pairs, and let $\varphi : \mathcal{C}(X) \to \mathcal{C}(Y)$ be a chain map such that*

$$\varphi(\mathcal{C}(A)) \subset \mathcal{C}(D).$$

Then the induced map $\bar{\varphi} : \mathcal{C}(X, A) \to \mathcal{C}(Y, D)$ given by the formula

$$\bar{\varphi}([c]_A) := [\varphi(c)]_D$$

is well defined and it is a chain map.

Proof. Is left as exercise. □

This leads to the following definition.

$$f : (X, A) \to (Y, D)$$

is a *continuous map between cubical pairs* if $f : X \to Y$ is continuous and $f(A) \subset D$.

To generate a map on the level of relative homology, i.e.

$$\bar{f}_* : H_*(X, A) \to H_*(Y, D).$$

we proceed as it was the case with $f_* : H_*(X) \to H_*(Y)$. In order to understand the definition, we must initially distinguish $\bar{f}_*$ from f_* but as we get used to this concept we will use f_* for either map and guess from the context if this is a map on relative homology.

Since $f : X \to Y$ is continuous, there exists an appropriate scaling vector α, such that $M_{f^\alpha} : X^\alpha \rightrightarrows Y$ is acyclic. Recall that the definition of $f_* : H_*(X) \to H_*(Y)$ involves scaling $\Lambda^\alpha : X \to X^\alpha$ and its inverse scaling $\Omega^\alpha : X^\alpha \to X$. In order to extend this definition to relative homology we must first know that all involved maps send pairs to pairs. Indeed, since A is cubical, $A^\alpha = \Lambda^\alpha(A)$ is cubical too, so we have a well defined map on pairs

$$\Lambda^\alpha : (X, A) \to (X^\alpha, A^\alpha).$$

Next, the acyclic representation M_{Λ^α} maps any elementary cube $Q \in \mathcal{K}(A)$ to $Q^\alpha \subset A^\alpha$ so its chain selector $\theta : \mathcal{C}(X) \to \mathcal{C}(X^\alpha)$ must send $\mathcal{C}(A)$ to $\mathcal{C}(A^\alpha)$. Therefore, by Proposition 9.9, we have a well defined map

$$\bar{\theta} : \mathcal{C}(X, A) \to \mathcal{C}(X^\alpha, A^\alpha)$$

and, consequently,

$$\bar{\Lambda}^\alpha_* := \bar{\theta}_* : H_*(X, A) \to H_*(X^\alpha, A^\alpha).$$

By the same reasoning we get the map

$$\bar{\Omega}^\alpha_* : H_*(X^\alpha, A^\alpha) \to H_*(X, A).$$

By the same arguments as those presented in the proof of Proposition 6.51, those two maps are isomorphisms and mutual inverses on relative homologies.

From the definition of f^α it follows that $f^\alpha(A^\alpha) \subset D$. The fact that, D and A^α are cubical implies that $M_{f^\alpha}(A^\alpha) \subset D$. Now let $\varphi : \mathcal{C}(X^\alpha) \to \mathcal{C}(Y)$ be a chain selector for M_{f^α}. For any $Q \in \mathcal{K}(A^\alpha)$, $|\varphi(\widehat{Q})| \subset M_{f^\alpha}(Q) \subset D$, and hence $\varphi(\mathcal{C}(A^\alpha)) \subset \mathcal{C}(D)$. Thus, φ induces a chain map between the relative chain complexes

$$\bar{\varphi} : \mathcal{C}(X^\alpha, A^\alpha) \to \mathcal{C}(Y, D).$$

We can finally define $\bar{f}_*$ as follows

Definition 9.10 Let $f : (X, A) \to (Y, D)$ be a continuous map between cubical pairs. Let the scaling α, the chain maps θ and φ be as in the preceding discussion. Then the map $\bar{f}_* : H_*(X, A) \to H_*(Y, D)$ *induced by f in relative homology* is given by the formula

$$\bar{f}_* := \bar{\theta}_* \bar{\varphi}_*.$$

The "bar" notation can be omitted for simplicity of presentation.

All important properties of the map f_* discussed in Section 6.4.2 carry over to maps induced in relative homology. In particular, we get the following generalization of Theorem 6.58:

Theorem 9.11 *Assume $f : (X, A) \to (Y, D)$ and $g : (Y, D) \to (Z, B)$ are continuous maps between cubical pairs. Then $(g \circ f) : (X, A) \to (Z, B)$ is a map between cubical pairs and*

$$\overline{(g \circ f)}_* = \bar{g}_* \circ \bar{f}_*$$

□

Theorem 9.12 (Excision IsomorphismTheorem) *Let (X, A) be a cubical pair. Let $U \subset A$ be open in X and representable. Then $X \setminus U$ is a cubical set and the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism*

$$i_* : H_*(X \setminus U, A \setminus U) \to H_*(X, A).$$

Proof. It follows from Proposition 6.3 that $X \setminus U$ is representable and from Proposition 6.4 that $X \setminus U$ is a cubical set.

First we will prove that

$$\mathcal{K}(X) \setminus \mathcal{K}(X \setminus U) \subset \mathcal{K}(A). \tag{9.5}$$

For this end take $Q \in \mathcal{K}(X) \setminus \mathcal{K}(X \setminus U)$. Then $Q \cap U \neq \emptyset$. By Proposition 2.15(iv) $\operatorname{cl} \mathring{Q} \cap U \neq \emptyset$. Since U is open in X we get $\mathring{Q} \cap U \neq \emptyset$. Hence in particular $\mathring{Q} \cap \operatorname{cl} U \neq \emptyset$. Therefore by Proposition 2.15(vi) $Q \subset \operatorname{cl} U \subset \operatorname{cl} A \subset A$. This proves (9.5).

For the inclusion map $j : X \setminus U \to X$ we have $M_j(Q) = Q$ for every $Q \in \mathcal{K}(X \setminus U)$. Thus, the inclusion map $\iota : \mathcal{C}(X \setminus U) \to \mathcal{C}(X)$ is a chain selector for M_j. Let $\pi : \mathcal{C}(X) \to \mathcal{C}(X, A)$ be the projection map. Take an element $[c]_A \in \mathcal{C}(X, A)$. Put $c' := \sum_{Q \in \mathcal{K}(X \setminus U)} c(Q)\widehat{Q}$. Then $\pi(\iota(c')) = [c']_A$. We have $c - c' = \sum_{Q \in \mathcal{K}(X) \setminus \mathcal{K}(X \setminus U)} c(Q)\widehat{Q}$ and by (9.5) $|c - c'| \subset A$. This shows that $\pi(\iota(c')) = [c]_A$, i.e. $\pi \circ \iota : \mathcal{C}(X \setminus U) \to \mathcal{C}(X, A)$ is surjective.

We will show that

$$\ker \pi \circ \iota = \mathcal{C}(A \setminus U).$$

Take $c \in \ker \pi \circ \iota$. This is equivalent to $c \in \mathcal{C}(X \setminus U) \cap \mathcal{C}(A)$, which happens if and only if $|c| \subset A \setminus U$, i.e. $c \in \mathcal{C}(A \setminus U)$. Therefore, by the Fundamental Epimorphism Theorem (Corollary 13.44) the map i_* is an isomorphism. □

Exercises

9.1 Let $\mathcal{C} = \{C_k, \partial_k\}$ be an abstract chain complex. Let $\mathcal{C}'$ be a chain subcomplex. Define $H_*(\mathcal{C}, \mathcal{C}')$.

9.2 Let $Q \in \mathcal{K}_q$ be an elementary cube. Prove that

$$H_k(Q, \operatorname{bd}(Q)) \cong \begin{cases} \mathbf{Z} & \text{if } k = q \\ 0 & \text{otherwise.} \end{cases}$$

9.3 Prove Proposition 9.9.

9.4 Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the map given by the matrix

$$f = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Let $X = [-4, 4] \times [4, 4]$, $A = [-4, 2] \times [4, 4] \cup [2, 4] \times [4, 4]$, $Y = f(X)$, and $D = A \cap f(A)$. Compute $H_*(X, A)$ and the map f_*.

9.2 Exact Sequences

Relative homology turns out to be a very powerful tool. However, if the reader solved Exercise 9.4, then it is clear that our ability to compute relative homology groups, is rather limited. Thus, we want to look for more efficient methods of computing relative homology groups. Given a pair of cubical sets (X, A), ideally, we would have a theorem which relates $H_*(X, A)$ to $H_*(X)$ and $H_*(A)$. As we shall see in Section 9.3 such a theorem exists, but before we can state it we need to develop some more tools in homological algebra.

From the algebraic point of view, homology begins with a chain complex $\{C_k, \partial_k\}$ which can be thought of as a sequence of abelian groups and maps

$$\ldots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \ldots$$

with the property that

$$\operatorname{im} \partial_{k+1} \subset \ker \partial_k.$$

A very special case of this is the following.

Definition 9.13 A sequence (finite or infinite) of groups and homomorphisms

$$\ldots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to \ldots$$

is *exact* at G_2 if

$$\operatorname{im} \psi_3 = \ker \psi_2.$$

It is an *exact sequence* if it is exact at every group. If the sequence has a first or last element, then it is automatically exact at that group.

Example 9.14 Let $\pi : \mathbf{Z} \to \mathbf{Z}_2$ be given by $\pi(n) := n \bmod 2$. Then the sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}_2 \longrightarrow 0$$

where $\times 2$ is multiplication by 2 is exact. Notice that two of the homomorphisms are not marked. This is because they are uniquely determined. Let us consider the issue of exactness at

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}.$$

Observe that the image of 0 is 0, therefore exactness is equivalent to the assertion that the kernel of multiplication by 2 is 0. This is true because $\times 2 : \mathbf{Z} \to \mathbf{Z}$ is a monomorphism We now check for exactness at

$$\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}_2.$$

Observe that the image of $\times 2$ are the set of even integers. This is precisely the kernel of π, since $\pi(n) = n \bmod 2 = 0$ if and only n is even. Hence the sequence is exact at $\mathbf{Z}$.

Finally, consider

$$\mathbf{Z} \xrightarrow{\pi} \mathbf{Z}_2 \longrightarrow 0.$$

Notice that $\pi(\mathbf{Z}) = \mathbf{Z}_2$, since $\pi(1) = 1$.

Since the sequence is exact at each group it is exact.

To develop our intuition concerning exact sequences we will prove a few simple lemmas.

Lemma 9.15 $G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\phi} 0$ *is an exact sequence if and only if* ψ_1 *is an epimorphism.*

Proof. ($\Rightarrow$) Assume that $G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\phi} 0$ is an exact sequence. Since $\phi : G_0 \to 0$, $\ker \phi = G_0$. By exactness, $\operatorname{im} \psi_1 = \ker \phi = G_0$, i.e. ψ_1 is an epimorphism.

($\Leftarrow$) If ψ_1 is an epimorphism, then $\operatorname{im} \psi_1 = G_0$. Since $\phi : G_0 \to 0$, $\ker \phi = G_0$. Therefore, $\operatorname{im} \psi_1 = \ker \phi$. $\square$

Lemma 9.16 $0 \xrightarrow{\phi} G_1 \xrightarrow{\psi_1} G_0$ *is an exact sequence if and only if* ψ_1 *is a monomorphism.*

Proof. ($\Rightarrow$) Assume that the sequence is exact. Clearly, $\operatorname{im}\phi = 0$, thus $\ker\psi_1 = 0$, which implies that ψ_1 is a monomorphism.

($\Leftarrow$) If ψ_1 is a monomorphism, then $\ker\psi_1 = 0$. Since, $\phi : 0 \to G_1$, $\operatorname{im}\phi = 0$. Therefore, $\operatorname{im}\phi = \ker\psi_1$. □

Similar arguments lead to the following result.

Lemma 9.17 *Assume that*

$$G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

is an exact sequence. Then the following are equivalent:

1. ψ_3 *is an epimorphism,*
2. ψ_2 *is the zero homomorphism,*
3. ψ_1 *is a monomorphism.*

Definition 9.18 A *short exact sequence* is an exact sequence of the form

$$0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0.$$

Stated as a definition, it may appear that a short exact sequence is a rather obscure notion. However, it appears naturally in many situations.

Example 9.19 Consider a cubical pair (X, A) and for each k the following sequence

$$0 \to C_k(A) \xrightarrow{\iota_k} C_k(X) \xrightarrow{\pi_k} C_k(X, A) \to 0 \tag{9.6}$$

where ι_k is the inclusion map and π_k is the quotient map. That this is a short exact sequence follows from simple applications of the previous lemmas. To begin with, ι_k is a monomorphism because it is an inclusion. Therefore, by Lemma 9.16

$$0 \to C_k(A) \xrightarrow{\iota_k} C_k(X)$$

is exact. Similarly, by definition of relative chains π_k is an epimorphism. Hence, Lemma 9.15 implies that

$$C_k(X) \xrightarrow{\pi_k} C_k(X, A) \to 0$$

is exact. So all that remains is to show that the sequence is exact at $C_k(X)$.

By definition the kernel of π_k is $C_k(A)$. Moreover, since ι_k is an inclusion, $\operatorname{im}\iota_k = C_k(A)$, i.e. $\operatorname{im}\iota_k = \ker\pi_k$.

The short exact sequence (9.6) is called the *short exact sequence of a pair.*

The short exact sequence (9.6) has a nice feature, defined below, which distinguishes it from the short exact sequence in Example 9.14.

Definition 9.20 A short exact sequence

$$0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0. \tag{9.7}$$

splits if there exists a subgroup $H \subset G_2$ such that

$$G_2 = \operatorname{im} \psi_3 \oplus H.$$

Note that the above equation is also equivalent to $G_2 = \ker \psi_2 \oplus H$.

Example 9.21 The short exact sequence of a pair splits because

$$C_k(X) = C_k(A) \oplus H = \iota(C_k(A)) \oplus H$$

where

$$H = \bigoplus_{Q \in \mathcal{K}_k(X) \setminus \mathcal{K}_k(A)} \mathbf{Z}\widehat{Q}.$$

The short exact sequence in Example 9.14 does not split because the middle group **Z** is cyclic so it cannot be decomposed as a direct sum of two nontrivial subgroups.

The existence of a complementing subgroup H does not always seem evident. There are characterizations of splitting in terms of left inverse of ψ_3 and right inverse of ψ_2 which we leave as exercises.

Exercises

9.5 Prove Lemma 9.17.

9.6 Let X be an acyclic cubical set. The associated chain complex induces a sequence of maps between the chains

$$\ldots \to C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \ldots.$$

Prove that this sequence is exact except at $C_0(X)$. Observe from this that homology measures how far a sequence induced by a chain complex is from being exact.

9.7 Consider the short exact sequence in (9.7). Show that the following conditions are equivalent.

(a) The sequence splits.
(b) ψ_3 has a left inverse, i.e. there exists $\pi : G_2 \to G_3$ such that $\pi\psi_3 = \operatorname{id}_{G_3}$.
(c) ψ_2 has a right inverse, i.e. there exists $\iota : G_1 \to G_2$ such that $\psi_3\iota = \operatorname{id}_{G_1}$.

9.3 The Connecting Homomorphism

In the previous section we defined the notion of an exact sequence and proved some simple lemmas. In this section we shall prove a

fundamental theorem of homological algebra. As a corollary we will answer the motivating question of how the relative homology groups are related to the homology groups of each space in the pair.

Definition 9.22 Let $\mathcal{A} = \{A_k, \partial_k^{\mathcal{A}}\}$, $\mathcal{B} = \{B_k, \partial_k^{\mathcal{B}}\}$, and $\mathcal{C} = \{C_k, \partial_k^{\mathcal{C}}\}$ be chain complexes. Let 0 denote the trivial chain complex, i.e. the chain complex in which each group is the trivial group. Let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be chain maps. The sequence

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

is a *short exact sequence of chain complexes* if for every k

$$0 \to A_k \xrightarrow{\varphi_k} B_k \xrightarrow{\psi_k} C_k \to 0$$

is a short exact sequence.

Theorem 9.23 *("Zig-zag Lemma") Let*

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. Then, for each k there exists a homomorphism

$$\partial_* : H_{k+1}(\mathcal{C}) \to H_k(\mathcal{A})$$

given on homology classes of cycles in $c \in \mathcal{C}$ by

$$\partial_*([c]) := \varphi_k^{-1}(\partial_{k+1}^{\mathcal{B}}(\psi_{k+1}^{-1}(c))) + \operatorname{im} \partial_k^{\mathcal{A}}, \tag{9.8}$$

such that

$$\ldots \to H_{k+1}(\mathcal{A}) \xrightarrow{\varphi_*} H_{k+1}(\mathcal{B}) \xrightarrow{\psi_*} H_{k+1}(\mathcal{C}) \xrightarrow{\partial_*} H_k(\mathcal{A}) \to \ldots$$

is a long exact sequence.

A word of caution about the formula 9.8 is due here before we start the proof. We do not assume that ψ and ϕ are invertible, $\psi^{-1}(c)$ denotes the inverse image. Thus the sum

$$\varphi_k^{-1}(\partial_{k+1}^{\mathcal{B}}(\psi_{k+1}^{-1}(c))) + \operatorname{im} \partial_k^{\mathcal{A}}$$

is the algebraic sum of two sets. A part of the proof will be showing that this formula makes sense. The map ∂_* defined by the formula 9.8 is called *connecting homomorphism.*

Proof. Before we begin the proof, observe that this is a purely algebraic result. Thus, the only information that we have is that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are chain complexes and that they are related by short exact sequences. The proof, therefore, can only consist of seeing how elements of the different groups are mapped around. To keep track of this it is useful to make use of the following commutative diagram.

$$
\begin{array}{ccccccccc}
 & & \downarrow \partial^{\mathcal{A}}_{k+2} & & \downarrow \partial^{\mathcal{B}}_{k+2} & & \downarrow \partial^{\mathcal{C}}_{k+2} & & \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{\varphi_{k+1}} & B_{k+1} & \xrightarrow{\psi_{k+1}} & C_{k+1} & \longrightarrow & 0 \\
 & & \downarrow \partial^{\mathcal{A}}_{k+1} & & \downarrow \partial^{\mathcal{B}}_{k+1} & & \downarrow \partial^{\mathcal{C}}_{k+1} & & \\
0 & \longrightarrow & A_{k} & \xrightarrow{\varphi_{k}} & B_{k} & \xrightarrow{\psi_{k}} & C_{k} & \longrightarrow & 0 \\
 & & \downarrow \partial^{\mathcal{A}}_{k} & & \downarrow \partial^{\mathcal{B}}_{k} & & \downarrow \partial^{\mathcal{C}}_{k} & & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{\varphi_{k-1}} & B_{k-1} & \xrightarrow{\psi_{k-1}} & C_{k-1} & \longrightarrow & 0 \\
 & & \downarrow \partial^{\mathcal{A}}_{k-1} & & \downarrow \partial^{\mathcal{B}}_{k-1} & & \downarrow \partial^{\mathcal{C}}_{k-1} & &
\end{array}
$$

Step 1: The Construction of ∂_*. The first step is to define the map

$$\partial_* : H_{k+1}(\mathcal{C}) \to H_k(\mathcal{A}).$$

Let $\gamma \in H_{k+1}(\mathcal{C})$. This means that there is some cycle $c \in C_{k+1}$ such that $\gamma = [c]$. Somehow, we need to map c to a cycle in A_k. Looking back at the commutative diagram there are two directions (left or down) we might try to go. The first is down, but $\partial^{\mathcal{C}}_{k+1} c = 0$ since c is a cycle. So the other option is to go left. Unfortunately, the map ψ_{k+1} goes in the wrong direction. However, by assumption

$$0 \longrightarrow A_{k+1} \xrightarrow{\varphi_{k+1}} B_{k+1} \xrightarrow{\psi_{k+1}} C_{k+1} \longrightarrow 0$$

is exact. Thus, $C_{k+1} = \operatorname{im} \psi_{k+1}$. Therefore, there exists $b \in B_{k+1}$ such that $\psi_{k+1}(b) = c$. In terms of the diagram we have the following

$$
\begin{array}{ccc}
b & & c \\
B_{k+1} & \xrightarrow{\psi_{k+1}} & C_{k+1} \\
 & & \downarrow \partial^{\mathcal{C}}_{k+1} \\
 & & C_k \\
 & & 0
\end{array}
$$

Of course, if $\ker \psi_{k+1} \neq 0$, then b is not uniquely defined. Let us ignore the non-uniqueness for the moment and continue our attempts to get to A_k. Given that $b \in B_{k+1}$ there are, again, two directions (left or down) we might try to go to get to A_k. Let's try to go down first, so we get $\partial^{\mathcal{B}}_{k+1} b \in B_k$. In order to get to A_k we need to ask whether $\partial^{\mathcal{B}}_{k+1} b \in \operatorname{im} \varphi_k$. Recall that

$$0 \longrightarrow A_k \xrightarrow{\varphi_k} B_k \xrightarrow{\psi_k} C_k \longrightarrow 0$$

is exact, so $\operatorname{im}\varphi_k = \ker\psi_k$. Therefore, $\partial^{\mathcal{B}}_{k+1}b \in \operatorname{im}\varphi_k$ if and only if $\partial^{\mathcal{B}}_{k+1}b \in \ker\psi_k$. Now recall that the big diagram commutes, so in particular the following sub-diagram commutes.

$$\begin{array}{ccccc}
b & & & & c \\
 & B_{k+1} & \xrightarrow{\psi_{k+1}} & C_{k+1} & \\
 & \downarrow \partial^{\mathcal{B}}_{k+1} & & \downarrow \partial^{\mathcal{C}}_{k+1} & \\
 & B_k & \xrightarrow{\psi_k} & C_k & \\
\partial b & & & & 0
\end{array}$$

But this means that

$$\psi_k\partial^{\mathcal{B}}_{k+1}b = \partial^{\mathcal{C}}_{k+1}\psi_{k+1}b = \partial^{\mathcal{C}}_{k+1}c = 0.$$

Thus $\partial^{\mathcal{B}}_{k+1}b \in \ker\psi_k = \operatorname{im}\varphi_k$. Therefore, there exists an $a \in A_k$ such that $\varphi_k a = \partial^{\mathcal{B}}_{k+1}b$. Moreover, such a is unique because the sequence is exact and hence φ_k is a monomorphism.

We have made it to A_k, but we have to verify that a is a cycle. Again we call on the commutativity of the big diagram to conclude that the following sub-diagram commutes

$$\begin{array}{cccccc}
 & & & & & b \\
 & & & & B_{k+1} & \\
 & & & & \downarrow \partial^{\mathcal{B}}_{k+1} & \\
 & & a & & & \\
0 & \longrightarrow & A_k & \xrightarrow{\varphi_k} & B_k & \partial b \\
 & & \downarrow \partial^{\mathcal{A}}_k & & \downarrow \partial^{\mathcal{B}}_k & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{\varphi_{k-1}} & B_{k-1} & \\
 & & \partial a & & & 0
\end{array}$$

and hence we have

$$\varphi_{k-1}\partial^{\mathcal{A}}_k a = \partial^{\mathcal{B}}_k\varphi_k a = \partial^{\mathcal{B}}_k\partial^{\mathcal{B}}_{k+1}b = 0.$$

Again by exactness φ_{k-1} is a monomorphism and therefore, $\partial^{\mathcal{A}}_k a = 0$.

In summary, starting with a cycle $c \in C_{k+1}$ we have produced a cycle $a \in A_k$ which is summarized in the following diagram.

$$\begin{array}{ccccc}
 & & b & & c \\
 & & B_{k+1} & \xrightarrow{\psi_{k+1}} & C_{k+1} \\
 & & \downarrow \partial^{\mathcal{B}}_{k+1} & & \\
A_k & \xrightarrow{\varphi_k} & B_k & & \\
a & & \partial b & &
\end{array} \tag{9.9}$$

So define

$$\partial_*([c]) := [a]. \tag{9.10}$$

In order to know that this map is well defined we must verify two things: First, as we mentioned before, b is not uniquely determined, so we have to check that the equivalence class $[a]$ is independent on the choice of b. Secondly, we must check that $[a]$ is independent on the choice of the representant c of $\gamma = [c]$. We will do the first verification while leaving the second one as exercise. So let $\bar{b} \in B_{k+1}$ be such that $\psi_{k+1}\bar{b} = c$. Before going further observe that,

$$\psi_{k+1}(b - \bar{b}) = \psi_{k+1}b - \psi_{k+1}\bar{b} = c - c = 0.$$

Therefore,

$$b - \bar{b} \in \ker \psi_{k+1} = \operatorname{im} \varphi_{k+1}$$

and there exists $q \in A_{k+1}$ such that $\varphi_{k+1}q = b - \bar{b}$.

Moreover let $\bar{a}$ be the unique element in A_k such that $\varphi_k \bar{a} = \partial^{\mathcal{B}}_{k+1}\bar{b}$. If ∂_* is independent on the choice of b then it must be that $[a] = [\bar{a}]$. Thus, we need to show that $a - \bar{a}$ is a boundary element. Again, we go back to the commutative diagram:

$$\begin{array}{ccccccc}
 & & q & & & & b - \bar{b} \\
 & & A_{k+1} & \xrightarrow{\varphi_{k+1}} & B_{k+1} & & \\
 & & \Big\downarrow \partial^{\mathcal{A}}_{k+1} & & \Big\downarrow \partial^{\mathcal{B}}_{k+1} & & \\
0 & \longrightarrow & A_k & \xrightarrow{\varphi_k} & B_k & & \\
 & & \partial q & & & & \partial(b - \bar{b})
\end{array}$$

Hence,

$$\varphi_k \partial^{\mathcal{A}}_{k+1} q = \partial^{\mathcal{B}}_{k+1}(b - \bar{b}) = \varphi_k(a - \bar{a}).$$

But, by exactness φ_k is a monomorphism, therefore $a - \bar{a} = \partial^{\mathcal{A}}_{k+1} q \in \operatorname{im} \partial^{\mathcal{A}}_{k+1}$. Finally, we leave it as exercise to verify that our map ∂_* is a homomorphism.

We now turn to the task of showing that the homology sequence is exact.

Step 2: Exactness at $H_k(\mathcal{B})$. We need to show that $\operatorname{im} \varphi_* = \ker \psi_*$ and will do it in two steps.

($\operatorname{im} \varphi_* \subset \ker \psi_*$) Let $\beta \in \operatorname{im} \varphi_*$. Then, there exists $\alpha \in H_k(\mathcal{A})$ such that $\varphi_* \alpha = \beta$. On the chain level, there is a cycle $a \in A_k$ such that $[a] = \alpha$. By definition $\beta = [\varphi_k a]$ and

$$\psi_* \beta := [\psi_k(\varphi_k a)].$$

By exactness $\psi_k \varphi_k = 0$, hence $\psi_* \beta = 0$.

($\ker \psi_* \subset \operatorname{im} \varphi_*$) Let $\beta \in \ker \psi_*$ and let $b \in B_k$ such that $\beta = [b]$. By assumption $0 = [\psi_k b]$, and hence $\psi_k b = \partial^{\mathcal{C}}_{k+1} c$ for some $c \in C_{k+1}$. By exactness, ψ_{k+1} is an epimorphism, so there exists $b' \in B_{k+1}$ such that $\psi_{k+1} b' = c$. Viewing this in terms of the commutative diagram we have

$$\begin{array}{ccccccccc}
 & & & & b' & & c & & \\
 & & & & B_{k+1} & \xrightarrow{\psi_{k+1}} & C_{k+1} & \longrightarrow & 0 \\
 & & & & \downarrow \partial^{\mathcal{B}}_{k+1} & & \downarrow \partial^{\mathcal{C}}_{k+1} & & \\
0 & \longrightarrow & A_k & \xrightarrow{\varphi_k} & B_k & \xrightarrow{\psi_k} & C_k & \longrightarrow & 0 \\
 & & & & b & & \partial c & &
\end{array}$$

though we cannot claim that $\partial^{\mathcal{B}}_{k+1} b' = b$.

However, we have

$$\psi_k(b - \partial^{\mathcal{B}}_{k+1} b') = \psi_k b - \psi_k(\partial^{\mathcal{B}}_{k+1} b') = 0.$$

This implies that $b - \partial^{\mathcal{B}}_{k+1} b' \in \ker \psi_k = \operatorname{im} \varphi_k$, by exactness. Therefore, there exists a unique (φ_k is a monomorphism) $a \in A_k$ such that $\varphi_k a = b - \partial^{\mathcal{B}}_{k+1} b'$. Moving down in the commutative diagram we have

$$\begin{array}{ccccccccc}
 & & a & & b - \partial b' & & \partial^{\mathcal{C}}_{k+1} c & & \\
0 & \longrightarrow & A_k & \xrightarrow{\varphi_k} & B_k & \xrightarrow{\psi_k} & C_k & \longrightarrow & 0 \\
 & & \downarrow \partial^{\mathcal{A}}_k & & \downarrow \partial^{\mathcal{B}}_k & & \downarrow \partial^{\mathcal{C}}_k & & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{\varphi_{k-1}} & B_{k-1} & \xrightarrow{\psi_{k-1}} & C_{k-1} & \longrightarrow & 0 \\
 & & & & 0 & & & &
\end{array}$$

Thus,

$$\varphi_{k-1} \partial^{\mathcal{A}}_k a = \partial^{\mathcal{B}}_k (b - \partial^{\mathcal{B}}_{k+1} b') = \partial^{\mathcal{B}}_k b - 0 = 0,$$

because by assumption b is a cycle. Since φ_{k-1} is a monomorphism, $\partial^{\mathcal{A}}_k a = 0$ and hence a is a cycle.

Finally,

$$\varphi_*[a] = [\varphi_k a] = [b - \partial^{\mathcal{B}}_{k+1} b'] = [b].$$

Therefore, $\beta = [b] \in \operatorname{im} \varphi_*$.

Step 3: Exactness at $H_k(\mathcal{C})$. Again the proof is in two steps.

($\operatorname{im} \psi_* \subset \ker \partial_*$) Assume $\gamma = [c] \in \operatorname{im} \psi_*$, i.e. $\gamma = [\psi_k b]$, where b is a cycle in B_k.

Let $a \in A_{k-1}$ be associated with c and b as in (9.9) and (9.10). We have $\varphi_{k-1} a = \partial^{\mathcal{B}}_k b = 0$. Since φ_{k-1} is a monomorphism, $a = 0$. Thus,

$$\partial_* \gamma = [a] = 0.$$

($\ker \partial_* \subset \operatorname{im} \psi_*$) Now assume that $0 = \partial_*[c] = [a]$. Thus, a is a boundary element, i.e. there exists $\bar{a} \in A_{k+1}$ such that $\partial^{\mathcal{A}}_{k+1} \bar{a} = a$. Expanding (9.9) one has

$$\begin{array}{ccccc}
\bar{a} & & b & & \\
A_k & \xrightarrow{\varphi_k} & B_k & \xrightarrow{\psi_k} & C_k \\
\downarrow \partial^{\mathcal{A}}_k & & \downarrow \partial^{\mathcal{B}}_k & & \\
A_{k-1} & \xrightarrow{\varphi_{k-1}} & B_{k-1} & & \\
a & & \partial b & &
\end{array}$$

Since the diagram commutes,

$$\partial_k^{\mathcal{B}} \varphi_k \bar{a} = \varphi_{k-1} \partial_k^{\mathcal{A}} a = \partial_k^{\mathcal{B}}.$$

This implies that

$$\partial_k^{\mathcal{B}}(b - \varphi_k \bar{a}) = \partial_k^{\mathcal{B}} b - \partial_k^{\mathcal{B}} \varphi_k \bar{a} = 0.$$

Thus, $b - \varphi_k \bar{a}$ is a cycle. Furthermore,

$$\psi_*([b - \varphi_k \bar{a}]) = [\psi_k(b - \varphi_k \bar{a})] = [\psi_k b - \psi_k \varphi_k \bar{a}] = [\psi_k b] = [c].$$

Therefore, $[c] \in \operatorname{im} \psi_*$.

Step 4: Exactness at $H_k(\mathcal{A})$. ($\operatorname{im} \partial_* \subset \ker \varphi_*$) Let $\alpha = [a] \in \operatorname{im} \partial_*$. Then there exists $\bar{a} \in A_{k+1}$ such that $\partial_{k+1}^{\mathcal{A}} \bar{a} = a$. Let $b = \varphi_{k+1} \bar{a}$. By definition and exactness at the chain level,

$$\varphi_* \alpha = [\varphi_{k-1} a] = [\varphi_{k-1} \partial_k^{\mathcal{A}} \bar{a}] = [\partial_k^{\mathcal{B}} b] = 0,$$

because $\partial_k^{\mathcal{B}} b$ is a boundary.

($\ker \varphi_* \subset \operatorname{im} \partial_*$) Assume $\varphi_* \alpha = 0$ and let $\alpha = [a]$. This implies that $\varphi_{k-1} a = \partial_k^{\mathcal{B}} b$ for some $b \in B_k$. Let $c = \psi_k b$. By exactness on the chain level,

$$\partial_k^{\mathcal{C}} c = \partial_k^{\mathcal{C}} \psi_k b = \psi_{k-1} \partial_k^{\mathcal{B}} b = \psi_{k-1} \varphi_{k-1} a = 0.$$

Observe that $\partial_*[c] = [a]$. Thus, $\alpha \in \operatorname{im} \partial_*$. □

While the proof may have seemed endless, this theorem is very powerful. For example, in the case of cubical pairs we have the following corollary.

Corollary 9.24 (The exact homology sequence of a pair) *Let* (X, A) *be a cubical pair. Then there is a long exact sequence*

$$\ldots \to H_{k+1}(A) \xrightarrow{\iota_*} H_{k+1}(X) \xrightarrow{\pi_*} H_{k+1}(X, A) \xrightarrow{\partial_*} H_k(A) \to \ldots$$

where $\iota : \mathcal{C}(A) \hookrightarrow \mathcal{C}(X)$ *is the inclusion map and* $\pi : \mathcal{C}(X) \to \mathcal{C}(X, A)$ *is the quotient map.*

Proof. The conclusion follows from Example 9.19 and Theorem 9.23. □

Example 9.25 To see how this result can be used consider $X = [-4, 4] \times [-4, 4]$ and $A = [-4, -2] \times [-4, 4] \cup [2, 4] \times [-4, 4]$. We want to compute $H_*(X, A)$. The long exact sequence for this pair is

$$\begin{aligned} \ldots \to H_2(A) &\xrightarrow{\iota_*} H_2(X) \xrightarrow{\pi_*} H_2(X, A) \xrightarrow{\partial_*} \\ H_1(A) &\xrightarrow{\iota_*} H_1(X) \xrightarrow{\pi_*} H_1(X, A) \xrightarrow{\partial_*} \\ H_0(A) &\xrightarrow{\iota_*} H_0(X) \xrightarrow{\pi_*} H_0(X, A) \xrightarrow{\partial_*} 0 \end{aligned} \tag{9.11}$$

X is a rectangle and hence acyclic. A is the disjoint union of two rectangles and therefore the disjoint union of two acyclic sets. Thus $H_0(X) = \mathbf{Z}\eta$ and

$H_0(A) = \mathbf{Z}\zeta_1 \oplus \mathbf{Z}\zeta_2$, where η, ζ_1, and ζ_2 are homology classes of arbitrarily chosen elementary vertices, respectively, in X, and in each component of A.

Furthermore, Proposition 9.3 implies that $H_0(X, A) = 0$. So we can rewrite (9.11) as

$$\begin{array}{rcccccl} \ldots \to & 0 & \xrightarrow{\iota_*} & 0 & \xrightarrow{\pi_*} & H_2(X,A) & \xrightarrow{\partial_*} \\ & 0 & \xrightarrow{\iota_*} & 0 & \xrightarrow{\pi_*} & H_1(X,A) & \xrightarrow{\partial_*} \\ & \mathbf{Z}\zeta_1 \oplus \mathbf{Z}\zeta_2 & \xrightarrow{\iota_*} & \mathbf{Z}\eta & \xrightarrow{\pi_*} & 0 & \xrightarrow{\partial_*} 0 \end{array}$$

By exactness $H_k(X, A) = 0$ for $k \geq 2$. Since ι is the inclusion map, and all vertices of A is are homologous in X, ι_* sends both ζ_1, and ζ_2 to η. Hence $\iota_* : \mathbf{Z}\zeta_1 \oplus \mathbf{Z}\zeta_2 \to \mathbf{Z}\eta$ is an epimorphism and $\ker \iota_* = \mathbf{Z}(\zeta_2 - \zeta_1) \cong \mathbf{Z}$. Again by exactness, $\partial_* : H_1(X, A) \to \mathbf{Z}\zeta_1 \oplus \mathbf{Z}\zeta_2$ is a monomorphism and $\operatorname{im} \partial_* = \ker \iota_* \cong \mathbf{Z}$. Therefore $H_1(X, A) \cong \mathbf{Z}$.

Relative homology provides necessary criterion for A to be a deformation retract of X.

Proposition 9.26 *If (X, A) is a cubical pair and A is a deformation retract of X, then $H_*(X, A) = 0$.*

Proof. Consider the following portion of the long exact sequence of a pair

$$\ldots \to H_k(A) \xrightarrow{\iota_*} H_k(X) \xrightarrow{\pi_*} H_k(X,A) \xrightarrow{\partial_*} H_{k-1}(A) \xrightarrow{\iota_*} H_{k-1}(X) \to \ldots$$

By Theorem 6.64, A and X have the same homotopy type. Thus, by Corollary 6.68 $H_*(A) \cong H_*(X)$. Furthermore, by the proof of Theorem 6.64 the inclusion map $\iota_* : H_*(A) \to H_*(X)$ is an isomorphism.

Returning to the exact sequence since ι_* is an isomorphism $\pi_*(H_k(X)) = 0$ and hence ∂_* is a monomorphism. However, since ι_* is an isomorphism $\partial_*(H_k(X, A)) = 0$ and thus $H_k(X, A) = 0$. □

Exercises

9.8 Let

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. Show that if any two of them has trivial homology, then so has the third one.

9.9 Complete the proof of Theorem 9.23 by showing that

(a) The construction of $\partial_*\gamma$ is independent on the choice of a cycle c such that $[c] = \gamma$;
(b) The map ∂_* is a homomorphism.

9.4 Mayer-Vietoris Sequence

Recall that Theorem 2.78 showed that given two acyclic sets whose intersection is acyclic can be combined to form a larger acyclic set. In this section we want to generalize this result to show how the homology of a set can be computed from subsets.

Theorem 9.27 *Let X be a cubical space. Let A_0 and A_1 be cubical subsets of X such that $X = A_0 \cup A_1$ and let $B = A_0 \cap A_1$. Then, there is a long exact sequence*

$$\ldots \to H_k(B) \to H_k(A_0) \oplus H_k(A_1) \to H_k(X) \to H_{k-1}(B) \to \ldots.$$

before we start the proof, we need the following lemma on direct sums of abstract chain complexes which is of interest by itself.

Lemma 9.28 *Let $(\mathcal{C}, \partial)$ and $(\mathcal{C}', \partial')$ be two chain complexes. Define the structure $(\mathcal{C} \oplus \mathcal{C}', \partial \oplus \partial')$ as follows: $\mathcal{C} \oplus \mathcal{C}'$ is a sequence of groups $C_k \oplus C'_k$ and $\partial \oplus \partial'$ is a sequence of homomorphisms $\partial \oplus \partial' : C_k \oplus C'_k \to C_{k-1} \oplus C'_{k-1}$ defined by*

$$(\partial \oplus \partial')(c, c') := (\partial c, \partial' c').$$

Then $(\mathcal{C} \oplus \mathcal{C}', \partial \oplus \partial')$ is a chain complex and

$$H_k(\mathcal{C} \oplus \mathcal{C}') \cong H_k(\mathcal{C}) \oplus H_k(\mathcal{C}').$$

Proof. For the simplicity of presentation we identify here the groups C_k and C'_k with the subgroups $C_k \times 0$ and, respectively, $0 \times C'_k$ of the direct sum $C_k \oplus C'_k$, defined via the product $C_k \times C'_k$ (see Chapter 13). This will make the proof a little sketchy so some details are left as exercise.

The verification that $\partial \oplus \partial'$ is a boundary map is obvious. Hence $\mathcal{C} \oplus \mathcal{C}'$ is a chain complex. It is also easy to verify that

$$\ker(\partial_k \oplus \partial'_k) = \ker \partial_k \oplus \ker \partial'_k$$

and

$$\operatorname{im}(\partial_k \oplus \partial'_k) = \operatorname{im} \partial_k \oplus \operatorname{im} \partial'_k$$

From this it is easy to verify that

$$H_k(\mathcal{C} \oplus \mathcal{C}') = \frac{\ker \partial_k \oplus \ker \partial'_k}{\operatorname{im} \partial_k \oplus \operatorname{im} \partial'_k} \cong \frac{\ker \partial_k}{\operatorname{im} \partial_k} \oplus \frac{\ker \partial'_k}{\operatorname{im} \partial'_k} = H_k(\mathcal{C}) \oplus H_k(\mathcal{C}').$$

□

Proof of Theorem 9.27: The proof consists of finding an appropriate short exact sequence of chain complexes and then applying Theorem 9.23. From Lemma 9.28, we see is that the short exact sequence we are looking for is

$$0 \to C_k(B) \xrightarrow{\varphi_k} C_k(A_0) \oplus C_k(A_1) \xrightarrow{\psi_k} C_k(X) \to 0 \tag{9.12}$$

for all k, where maps φ_k and ψ_k have to be defined. Clearly, $B \subset A_0$ and $B \subset A_1$, therefore $\mathcal{C}(B) \subset \mathcal{C}(A_0)$ and $\mathcal{C}(B) \subset \mathcal{C}(A_1)$. For any chain $c \in \mathcal{C}(B)$ define

$$\varphi_k c := (c, -c).$$

Similarly, $C_k(A_0) \subset cC_k(X)$ and $C_k(A_1) \subset C_k(X)$, so we may define

$$\psi_k(d, e) := d + e.$$

The verification that φ and ψ are chain maps is obvious. We need to show that the sequence (9.12) is exact.

We begin by checking exactness at $C_k(B)$. This is equivalent to showing that φ_k is a monomorphism. But $\varphi_k c = (c, -c) = 0$ implies that $c = 0$.

Exactness at $C_k(A_0) \oplus C_k(A_1)$ is slightly more complicated. Observe that

$$\psi_k \varphi_k(c) = \psi_k(c, -c) = c + (-c) = 0.$$

This implies that $\operatorname{im} \varphi_k \subset \ker \psi_k$. To prove the other inclusion, assume that

$$0 = \psi_k(d, e) = d + e.$$

Thus, $d = -e$. In particular, $d \in C_k(A_0) \cup C_k(A_1) = C_k(B)$. Then $\varphi_k(d)$ is well defined and $\varphi_k(d) = (d, -d) = (d, e)$, so $(d, e) \in \operatorname{im} \varphi_k$.

Finally, it needs to be checked that (9.12) is exact at $C_k(X)$. Let $c \in C_k(X)$. Then by Proposition 2.77 $c = c_0 + c_1$ for some $c_0 \in C_k(A_0)$ and $c_1 \in C_k(A_1)$. But

$$\psi_k(c_0, c_1) = c$$

and therefore $c \in \operatorname{im} \psi_k$.

Since (9.12) is an exact sequence, the long exact sequence exists by Theorem 9.23. □

The Mayer-Vietoris theorem provides us with an excellent tool to compute homology groups of interesting spaces.

Recall that in examples and exercises of Chapter 2 a fair amount of effort was spent on studying the set Γ^d, the boundary of unit cube $[0,1]^{d+1}$, in case of $d = 1$ and $d = 2$. The direct computation of $H_k(\Gamma^d)$ increased in complexity as d increased. For $d = 0$, the result is obvious because then $\Gamma^0 = \{0, 1\}$ is of dimension 0, so the only nontrivial homology group is

$$H_0(\Gamma^0) = C_0(\Gamma^0) = \mathbf{Z}\widehat{[0]} \oplus \mathbf{Z}\widehat{[1]} \cong \mathbf{Z} \oplus \mathbf{Z}.$$

When $d = 1$, we were able to compute the homology by hand in Example 2.47 but already for $d = 2$ the direct computation seemed endless and we were tempted to apply Homology program . Getting the formula for an arbitrary n seemed beyond our technical abilities. We can now derived it from the Mayer-Vietoris sequence.

Theorem 9.29 *Consider* $\Gamma^d = \text{bd}\,[0,1]^{d+1}$*, where* $d \geq 1$*. Then*

$$H_k(\Gamma^d) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0 \text{ or } k = d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since Γ^d is connected for $d \geq 1$, we know that $H_0(\Gamma^d) \cong \mathbf{Z}$. The formula for $H_k(\Gamma^d)$ will be derived for $k > 0$ by induction on d. By Example 2.47, we know the result is true for $d = 1$.

So assume that $d \geq 2$ and that the result it is true for Γ^k, $k = 1, \ldots, d-1$, we will show it is true for Γ^d. Consider the $(d-1)$-dimensional face

$$P := [0] \times [0,1]^{d-1}$$

of Γ^d and let

$$X' := \Gamma^d \setminus \mathring{P}$$

be the union of complementing faces. By definition, $P \cup X = \Gamma^d$. By Theorem 2.76, P is acyclic. Note that X' is precisely the image of $Q = [0,1]^{d+1}$ under the elementary collapse of Q by P, so by Theorem 2.68 it is acyclic. Furthermore, the cubical set

$$Y := P \cap X' = [0] \times \Gamma^{d-1}$$

is homeomorphic to Γ^{d-1} so we know its homology by the induction hypothesis.

We now use the Mayer-Vietoris sequence with $X = \Gamma^d$, $A_0 = P$, $A_1 = X'$, and $B = Y$, to compute $H_k(\Gamma^d)$.

The case $k = 1$ is actually the hardest one so we start from the highest dimension $k = d$. The portion of the sequence which is of our interest

$$\to H_d(P) \oplus H_d(X') \to H_d(\Gamma^d) \to H_{d-1}(Y) \to H_{d-1}(P) \oplus H_{d-1}(X') \to$$

Since P and X' are acyclic and $d \geq 2$, this reduces to

$$\to 0 \oplus 0 \to H_d(\Gamma^d) \to H_{d-1}([0] \times \Gamma^{d-1}) \to 0 \oplus 0 \to 0 \to$$

By exactness and by the induction hypothesis,

$$H_d(\Gamma^d) \cong H_{d-1}([0] \times \Gamma^{d-1}) \cong H_{d-1}(\Gamma^{d-1}) \cong \mathbf{Z}.$$

We leave it to the reader to check that $H_k(\Gamma^d) = 0$ for $1 \leq k < d$ and for $k > d$.

If $k = 1$, then the portion of the sequence which interests us is

$$\to H_1(P) \oplus H_1(X') \to H_1(\Gamma^d) \to H_0(Y) \to H_0(P) \oplus H_0(X') \to H_0(\Gamma^d) \to 0.$$

Again by the acyclicity, $H_1(P) = H_1(X') = 0$ so by exactness the map

$$H_1(\Gamma^d) \to H_0(Y)$$

is a monomorphism. On the other hand, the map

$$H_0(Y) \to H_0(P) \oplus H_0(X')$$

is induced by the map

$$\varphi_0 : C_0(Y) \to C_0(P) \oplus C_0(X')$$

given by $\varphi(c) = (c, -c)$. Since Y, P and X' are connected, the homology group of each is generated by the homology class of any chosen elementary vertex. So let us choose $V = [0] \in Y \cap P \cap X'$. Then

$$\varphi_*([\widehat{V}]) = ([\widehat{V}], -[\widehat{V}])$$

which is a monomorphism. By exactness,

$$H_1(\Gamma^d) \to H_0(Y)$$

is the zero map but we proved that it is a monomorphism. This forces

$$H_1(\Gamma^d) = 0.$$

□

Exercises

9.10 Consider the d-dimensional unit sphere

$$S_0^d := \{x \in \mathbf{R}^{d+1} \mid ||x||_0 = 1\}$$

where $||x||_0$ is the supremum norm (see Chapter 12). Note that S_0^d is a cubical set. By using the homeomorphism in Exercise 12.18 and Corollary 6.68 one can can conclude that S_0^d has the same homology groups as Γ^d.

(a) Derive this result directly by applying the Mayer-Vietoris sequence to

$$X := S_0^d,\ A_0 := \{x \in S_0^d \mid x_1 \leq 0\},\ A_1 := \{x \in S_0^d \mid x_1 \geq 0\},$$

and $B := A_0 \cap A_1 = [0] \times S_0^{d-1}$.

(b) Show that the reduced homology groups of spheres are

$$\tilde{H}_k(S_0^d) \cong \begin{cases} \mathbf{Z} & \text{if } k = d \\ 0 & \text{otherwise.} \end{cases}$$

9.11 In Exercise 1.12 you were asked to make numerical experiments and give a conjecture about homology of a graph G^n with n vertices where every pair of distinct vertices is connected by one edge. Now you are in position to prove your conjecture. More precisely, use the Mayer-Vietoris sequence to derive a formula for the dimension of $H_1(G^n)$ by induction on n.

9.5 Weak Boundaries

Recall that in Chapter 3 we mentioned that, given cubical sets X and Y, it would be desirable that our algorithm outputs $H_*(X)$ and $H_*(Y)$ in such a way that it is evident whether or not they are isomorphic. Did we achieve this goal? The output of our algorithm consists of two matrices and two numbers

$$(U, B, s, t)$$

described in the proof of Theorem 3.56. In our context, when the groups are $G := Z_k$ and $H := B_k$, the meaning of this output is the following:

1. The matrix B is the Smith normal form of the inclusion map $B_k \hookrightarrow Z_k$,
2. The columns of U are canonical coordinates of the new basis $\{u_1, u_2, \dots u_m\}$ of Z_k.
3. The nontrivial columns of BU are canonical coordinates of the new basis $\{b_1 u_1, b_2 u_2 \dots b_t u_t\}$ of B_k, where $b_i := B(i,i)$ are the diagonal entries of B,
4. The number s tells us that $b_i = 1$ if and only if $i \leq s$.
5. The number t tells us that $b_i = 0$ if and only if $i > t$.
6. The quotient group decomposition is

$$H_k(\mathcal{C}) = \bigoplus_{i=s+1}^{t} \langle [u_i] \rangle \;\oplus\; \bigoplus_{i=t+1}^{m} \mathbf{Z}[u_i] \tag{9.13}$$

If we only care about the identification of $H_k(\mathcal{C})$ up to isomorphism, all necessary information is contained in the matrix B. Indeed,

$$H_k(\mathcal{C}) \cong \mathbf{Z}^r \;\oplus\; \bigoplus_{i=s+1}^{t} \mathbf{Z}_{b_i},$$

where $r := m - t$, b_i are diagonal entries of B, $b_i > 1$ for $i = s+1, s+2, \dots t$, and $b_i | b_{i+1}$.

We shall push a little further the analysis of $H_k(\mathcal{C})$. The decomposition in (9.13) can be expressed as

$$H_k(\mathcal{C}) = T_k \oplus F_k$$

where

$$T_k := \bigoplus_{i=s+1}^{t} \langle [u_i] \rangle \cong \bigoplus_{i=s+1}^{t} \mathbf{Z}_{b_i}. \tag{9.14}$$

is its torsion subgroup, and its complement

$$F_k := \bigoplus_{i=t+1}^{m} \mathbf{Z}[u_i]$$

is a maximal free subgroup. Recall that T_k is independent on a choice of basis but its complement F_k in the above decomposition is not (see Exercises **??** and 3.13). Notice that we have isomorphisms

$$H_k(\mathcal{C})/T_k \cong F_k \cong \bigoplus_{i=t+1}^{m} \mathbf{Z}u_i \cong \mathbf{Z}^r. \tag{9.15}$$

On the other hand, if we only care about identification of our free component of the homology group, we can achieve the same goal by taking the quotient

$$Z_k/W_k$$

where

$$W_k := \bigoplus_{i=1}^{t} \mathbf{Z}u_i \subset Z_k. \tag{9.16}$$

Indeed, since W_k is generated by a subset of the basis $\{u_i\}$ of Z_k, we get

$$Z_k/W_k \cong \bigoplus_{i=t+1}^{m} \mathbf{Z}u_i \cong H_k(\mathcal{C})/T_k. \tag{9.17}$$

The group W_k is called *group of weak boundaries* of C_k. This name is justified by the following definition and proposition.

Definition 9.30 *A chain $c \in C_k$ is called* weak boundary *if there exists a non-zero integer β such that $\beta c \in B_k$.*

Proposition 9.31 *The group $W_k \subset Z_k$ given by the formula (9.16) is the set of all weak boundaries of C_k. In particular, the formula (9.16) is independent of the choice of basis $\{u_i\}$ of Z_k related to the Smith normal form B.*

Proof. If $c \in W_k$ then $c = \sum_{i=s}^{t} \alpha_i u_i$, so $\beta c \in B_k$ for $\beta := b_t b_{t+1} \dots b_s$, because $b_i u_i \in B_k$. Hence c is a weak boundary.

Conversely, suppose that $c \in C_k$ is a weak boundary and let $\beta \neq 0$ be such that $\beta c \in B_k$. First, note that $c \in Z_k$. Indeed, $B_k \subset Z_k$ hence $\beta(\partial c) = \partial(\beta c) = 0$. But this implies that $\partial c = 0$. Hence c can be expressed as a linear combination $c = \sum_{i=1}^{m} \alpha_i u_i$. Then

$$B_k \ni \beta c = \sum_{i=0}^{m} \beta \alpha_i u_i$$

This implies that $\beta \alpha_i = 0$ for all $i > t$. Since $\beta \neq 0$, we get $\alpha_i = 0$ for all $i > t$ and

$$c = \sum_{i=1}^{t} \alpha_i u_i \in W_k$$

The last conclusion is obvious because the definition of weak boundary is not related to any particular basis. □

Again a word of caution: All three groups in the formula (9.17) are isomorphic but, as it was pointed out before, the group in the middle is dependent on a choice of basis while the quotient groups on the left and right are independent.

Example 9.32 Consider the abstract chain complex studied in Example 3.60. The group $Z_0 = C_0$ is generated by all three columns $\{Q_1, Q_2, Q_3\}$ of the inverse change of coordinates transfer matrix Q while the group of weak boundaries W_0 is generated by $\{Q_1, Q_2\}$. Note that the second column Q_2 has the property $Q_2 \notin B_0$ but $2Q_2 \in B_0$. We get

$$H_0(\mathcal{C})/T_0 \cong Z_0/W_0 \cong \mathbf{Z}Q_3 \cong \mathbf{Z}.$$

The group $H_1(\mathcal{C})$ is free with $W_1 = B_1 = 0$, so its free component does not require any computation.

Exercises

10

Nonlinear Dynamics

This is perhaps the most challenging chapter of this book in that we attempt to show how homology can be used in nonlinear analysis and dynamical systems. As was mentioned in the Preface algebraic topology was developed to solve problems in these subjects. Thus, on the one hand some of the material (most notably Sections 10.4 and 10.5) is classical and has a rich history. On the other hand, the focus of Section 10.6, combining numerical analysis with homology via the computer to obtain mathematically rigorous results, is a cutting edge topic. We assume that the background and interest of our readers is equally varied; ranging from the an individual with no background in dynamics hoping to learn the mathematical theory to others who's primary goal is to understand how to apply these ideas to specific nonlinear systems.

In order to accommodate this variety we have tried to make this chapter as modular as possible. For the reader with no background in dynamics we recommend beginning with Sections 10.1 and 10.2 where, in as efficient a manner as possible, we have tried to introduce the essential ideas of the subject and indicate the potential complexity of nonlinear systems. Section 10.3 introduces the Ważewski principle which is perhaps the most straightforward example of how nontrivial topology can be used to draw conclusions about dynamics. We recommend this section to all readers.

The most simply stated question (and for many applications the most important) about a continuous map $f : X \to X$ is whether it has a fixed point, i.e. a point $x \in X$ such that $f(x) = x$. This problem has received an enormous amount of attention. In Sections 10.4 and 10.5 we discuss two fundamental techniques, the Lefschetz fixed point theorem and degree theory.

Readers who are more interested in complex or chaotic dynamics can immediately turn to Sections 10.6 and 10.7 at the end of the Chapter. The material of this section is written for the individual who wants to verify chaotic dynamics for specific problems.

As we saw in Section 6.5, homology has the property that if two maps or two spaces are homotopic then their homologies are the same. The converse is not true: there are spaces (though rather pathological) with the same homologies which are not homotopic, and there are non-homotopic maps which induce the same homology. Thus, the algebraic tools that we developed in this book are rather vague measurements of the topology. On the other hand, this is exactly this vagueness or forgetfulness of homology which allows to distinguish the most important properties of spaces and maps we care about, while reducing the infinite and even uncountable amount of information coming from topology to a finite information contained in algebra of cubical chain complexes and chain maps.

One of the goals of this chapter is to show that the algebraic topological tools being developed can be combined with standard numerical techniques to provide rigorous proofs about the dynamics of nonlinear systems.

10.1 Maps and Symbolic Dynamics

The simplest way to generate a dynamical system is to begin with a continuous map $f : X \to X$ defined on a metric space and iterate. In this setting X is referred to as the *phase space*. Given a point $x \in X$ its *forward orbit* is defined to be

$$\gamma^+(x, f) := \{f^n(x) \mid n = 0, 1, 2, \ldots\} \subset X.$$

Observe that the forward orbit of a point defines a subset of X. Unfortunately, forward orbits need not have particularly nice properties with respect to f. For example, while obviously $f(\gamma^+(x, f)) \subset \gamma^+(x, f)$, in general $f(\gamma^+(x, f)) \neq \gamma^+(x, f)$. To rectify this we would also like to consider the *backward orbit* of x_0, but since we have not assumed that f is invertible it makes no sense to write f^{-n}. The following definition circumvents this technicality.

Definition 10.1 Let $f : X \to X$ be a continuous map and let $x \in X$. A *full solution* through x under f is a function $\gamma_x : \mathbf{Z} \to X$ satisfying the following two properties:

1. $\gamma_x(0) = x$
2. $\gamma_x(n + 1) = f(\gamma_x(n))$ for all $n \in \mathbf{Z}$.

A simple warning: as the following two examples indicate, full solutions need not exist and even if they do they need not be unique. However, if f is invertible, then there exists a unique solution $\gamma_x(n) = f^n(x)$ for all $n \in \mathbf{Z}$.

Example 10.2 Consider the map $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. Observe that any negative number fails to have a pre-image. Therefore, if $x < 0$, then there does not exist a full solution γ_x.

Example 10.3 Consider the continuous map $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}; \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Observe that $f(0) = 0$, therefore there is an obvious full solution $\gamma_0 : \mathbf{Z} \to [0,1]$ through 0 defined by $\gamma_0(n) = 0$. However, we can also define another full solution $\gamma_0' : \mathbf{Z} \to [0,1]$ by

$$\gamma_0'(n) = \begin{cases} 0 & \text{if } n \geq 0; \\ 2^{n+1} & \text{if } n < 0. \end{cases}$$

Let $\gamma_x : \mathbf{Z} \to X$ be a full solution. The associated *orbit* of x is defined to be the set

$$\gamma_x(\mathbf{Z}) := \{\gamma_x(n) \mid n \in \mathbf{Z}\}.$$

Observe that $f(\gamma_x(\mathbf{Z})) = \gamma_x(\mathbf{Z})$. Thus, orbits are invariant with respect to the dynamics. The following definition generalizes this observation.

Definition 10.4 Let $f : X \to X$ be a continuous map. A set $S \subset X$ is an *invariant set* under f if

$$f(S) = S.$$

One of the goals of dynamical systems is to understand the existence and structure of invariant sets. In general this is an extremely difficult problem, in part because of the tremendous variety of types of orbits and combinations of orbits that can occur. To emphasise this point we begin our discussion with a very particular type of dynamical system known as symbolic dynamics. We will employ these systems in two ways. In this section, we will use them to rapidly introduce a variety of concepts from dynamical systems. Then, at the end of this chapter we will use them as models for other potentially more complicated systems.

We begin by defining the phase space of a symbolic dynamical system.

Definition 10.5 The *symbol space* on n symbols is given by

$$\begin{aligned} \Sigma_n &= \{1, \dots n\}^{\mathbf{Z}} \\ &= \{\mathbf{a} = (\dots a_{-1}, a_0, a_1, \dots) \mid a_j \in \{1, \dots, n\} \text{ for all } j \in \mathbf{Z}\}. \end{aligned}$$

The metric on Σ_n is defined to be

$$\operatorname{dist}(\mathbf{a}, \mathbf{b}) := \sum_{j=-\infty}^{\infty} \frac{\delta(a_j, b_j)}{4^{|j|}}$$

where

$$\delta(t, s) = \begin{cases} 0 & \text{if } t = s; \\ 1 & \text{if } t \neq s. \end{cases}$$

Before continuing our development of symbolic dynamics there are a couple of useful observations to be made. Notice that if $\mathbf{a}, \mathbf{b} \in \Sigma_n$ and $a_k \neq b_k$, then

$$\operatorname{dist}(\mathbf{a}, \mathbf{b}) \geq 4^{-|k|}. \tag{10.1}$$

On the other hand, if $a_j = b_j$ for all $|j| \leq k$, then

$$\operatorname{dist}(\mathbf{a},\mathbf{b}) \le 2 \sum_{j=k+1}^{\infty} \frac{1}{4^j}$$

and summing up the geometric series we get

$$\operatorname{dist}(\mathbf{a},\mathbf{b}) \le \frac{2}{3} 4^{-k}. \tag{10.2}$$

Proposition 10.6 *Σ_n is compact.*

Proof. By definition Σ_n is compact if every sequence in Σ_n contains a convergent subsequence. Thus, given an arbitrary infinite sequence $\left(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots\right)$ of elements of Σ_n. we need to find a convergent subsequence. Let

$$\mathbf{S} := \left\{\mathbf{a}^k \in \Sigma_n \mid k = 0, 1, 2, \dots\right\},$$

that is $\mathbf{S}$ is the set of elements of the original sequence. Observe that if $\mathbf{S}$ consists of a finite set of elements, then there must be at least one element $\mathbf{b} \in \Sigma_n$ which appears infinitely often in the sequence $\left(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots\right)$. Choose the subsequence consisting of those terms $\mathbf{a}^k$ which are equal to $\mathbf{b}$. Since this is a constant sequence, it converges to $\mathbf{b} \in \Sigma_n$. Having eliminated that case, we may assume now that $\mathbf{S}$ is an infinite set and that each term of the sequence $\left(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots\right)$ is repeated at most finitely many times. By eliminating the repetitions, we obtain an infinite subsequence in which no term is repeated. Therefore we may assume, without a loss of generality, that the terms of $\left(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots\right)$ are pairwise distinct.

For the remaining case we will construct an infinite collection of sets and positive integers which will be used to define a convergent subsequence.

Each element $\mathbf{a}^k \in \mathbf{S}$ takes the form

$$\mathbf{a}^k = (\dots, a_{-1}^k, a_0^k, a_1^k, \dots).$$

By definition, for each $k \in \mathbf{Z}$, $a_0^k \in \{1, \dots, n\}$. In particular, there are only a finite number of choices for the a_0^k. Therefore in the infinite sequence of integers a_0^k there must be at least one value which is repeated infinitely often. Let us call this value $b(0)$.

Define

$$\mathbf{S}^{(0)} := \left\{\mathbf{a}^k \in \mathbf{S} \mid a_0^k = b(0)\right\}.$$

Clearly, $\mathbf{S}^{(0)}$ is an infinite set. Let us also define $m(0) := \min\left\{k \mid a_0^k = b(0)\right\}$.

With this construction as a model we will now inductively define sets $\mathbf{S}^{(p)}$, positive integers $m(p)$, and integers $b(p) \in \{1, \dots, n\}$ which satisfy the following properties for every $p = 1, 2, \dots$,

1. $\mathbf{S}^{(p)} \subset \mathbf{S}^{(p-1)}$ and contains an infinite number of elements.
2. If p is even and $\mathbf{a}^k \in \mathbf{S}^{(p)}$, then $a_i^k = b(i)$ for $i = -\frac{p}{2}, \dots, \frac{p}{2}$. If p is odd and $\mathbf{a}^k \in \mathbf{S}^{(p)}$, then $a_i^k = b(i)$ for $i = -\frac{p-1}{2}, \dots, \frac{p+1}{2}$.

3. If p is even, then $m(p) \geq \min\left\{k \mid \mathbf{a}^k \in \mathbf{S}^{(p)},\ a^k_{-\frac{p}{2}} = b(p)\right\}$. If p is odd, then $m(p) \geq \min\left\{k \mid \mathbf{a}^k \in \mathbf{S}^{(p)},\ a^k_{\frac{p+1}{2}} = b(p)\right\}$. Furthermore, $m(p) > m(p-1)$.

Let us assume that the sets $\mathbf{S}^{(p)}$, positive integers $m(p)$, and integers $b(p) \in \{1, \dots, n\}$ have been defined for $p = \{0, 1, \dots, q\}$. We will now construct $\mathbf{S}^{(q+1)}$, $m(q+1)$ and $b(q+1)$.

To begin consider the case where q is odd. The set $\mathbf{S}^{(q)}$ contains an infinite number of elements, however $\left\{a^k_{-\frac{q+1}{2}} \mid a^k \in \mathbf{S}^{(q)}\right\} \subset \{1, \dots, n\}$ which is finite. Thus, we can choose an integer $b(q+1) \in \{1, \dots, n\}$ such that set

$$\mathbf{S}^{(q+1)} := \left\{\mathbf{a}^k \in \mathbf{S}^{(q)} \mid a^k_{-\frac{q+1}{2}} = b(q+1)\right\}$$

has an infinite number of elements. Observe that $\mathbf{S}^{(q+1)} \subset \mathbf{S}^{(q)}$. Obviously $q+1$ is even. By construction and the induction hypothesis if $\mathbf{a}^k \in \mathbf{S}^{(q+1)}$, then $a^k_i = b(i)$ for $i = -\frac{q+1}{2}, \dots, \frac{q+1}{2}$. Finally, define

$$m(q+1) := \min\left\{k \mid k > m(q),\ \mathbf{a}^k \in \mathbf{S}^{(q+1)},\ a^k_{-\frac{q+1}{2}} = b(q+1)\right\}.$$

The reader can check that $m(q+1)$ satisfies the desired properties. The construction in the case that q is odd is similar.

We are finally in the position to prove the existence of a convergent subsequence. Let $\mathbf{b} = (b_i) \in \Sigma_n$ be the sequence

$$b_i = \begin{cases} b(0) & \text{if } i = 0; \\ b(2i) & \text{if } i < 0; \\ b(2i-1) & \text{if } i > 0. \end{cases}$$

Consider the sequence $\left\{\mathbf{a}^{m(p)} \mid p = 0, 1, 2, \dots\right\}$. This is clearly a subsequence of the original sequence. We will now argue that this sequence coverges to $\mathbf{b}$. Since $\mathbf{a}^{m(p)} \in \mathbf{S}^{m(p)}$ if p is even, $a^{m(p)}_i = b(i)$ for $i = -\frac{p}{2}, \dots, \frac{p}{2}$, and if p is odd then $a^{m(p)}_i = b(i)$ for $i = -\frac{p-1}{2}, \dots, \frac{p+1}{2}$. Thus, by (10.2)

$$\operatorname{dist}\left(\mathbf{a}^{m(p)}, \mathbf{b}\right) < \frac{2}{3} 4^{-p+1}$$

which shows that $\lim_{p \to \infty} \mathbf{a}^{m(p)} = \mathbf{b}$. $\square$

Now that the phase space is defined we are ready to introduce the dynamics. The *shift map* on n symbols, $\sigma : \Sigma_n \to \Sigma_n$, is defined by

$$(\sigma(\mathbf{a}))_k = a_{k+1}.$$

Before we can treat this as a dynamical system we need to check that the shift map is continuous.

Proposition 10.7 *$\sigma : \Sigma_n \to \Sigma_n$ is a continuous function.*

Proof. We need to show that given $\epsilon > 0$ there exists $\delta > 0$ such that if $\operatorname{dist}(\mathbf{a}, \mathbf{b}) < \delta$, then $\operatorname{dist}(\sigma(\mathbf{a}), \sigma(\mathbf{b})) < \epsilon$. So fix $\epsilon > 0$. Then there exists $k > 0$ such that $\frac{2}{3}4^{-k} < \epsilon$. Now take $\mathbf{a}, \mathbf{b} \in \Sigma_n$ such that $\operatorname{dist}(\mathbf{a}, \mathbf{b}) < \delta := 4^{-(k+1)}$. By (10.1) we know that this implies that $a_j = b_j$ for all $|j| \leq k+1$. Now observe that $\sigma(\mathbf{a})_j = \sigma(\mathbf{b})_j$ for all $|j| \leq k$. Then, by (10.2), $\operatorname{dist}(\sigma(\mathbf{a}), \sigma(\mathbf{b})) < \epsilon$. □

The importance of symbolic dynamics is that it is extremely easy to identify individual orbits with particular properties. As such, we can use it to provide concrete examples for a variety of important definitions. In what follows $f : X \to X$ will always denote a continuous map.

Definition 10.8 An element $x \in X$ is a *fixed point* of f, if $f(x) = x$.

Example 10.9 Consider $\sigma : \Sigma_2 \to \Sigma_2$. There are exactly two fixed points:

$$(\ldots 1, 1, 1, 1, 1, \ldots) \quad \text{and} \quad (\ldots 2, 2, 2, 2, 2, \ldots).$$

Definition 10.10 Let $y, z \in X$ be fixed points for f. We say that x is a *heteroclinic point* from y to z if there exists a full solution γ_x such that

$$\lim_{n\to\infty} f^n(x) = z \quad \text{and} \quad \lim_{n\to-\infty} \gamma_x(n) = y.$$

The associated orbit is called a *heteroclinic orbit*. The point x is *homoclinic* if $y = z$.

Example 10.11 If $\mathbf{a} \in \Sigma_3$ is defined by

$$a_j = \begin{cases} 2 & \text{if } j < 0; \\ 3 & \text{if } j = 0; \\ 1 & \text{if } j > 0, \end{cases}$$

then $\mathbf{a}$ is a heteroclinic point from $(\ldots, 2, 2, 2, \ldots)$ to $(\ldots, 1, 1, 1, \ldots)$. To verify this statement we need to show that $\lim_{n\to\infty} \sigma^n(a) = (\ldots, 1, 1, 1, \ldots)$. Observe that by (10.2)

$$\operatorname{dist}(\sigma^k(\mathbf{a}), (\ldots, 1, 1, 1, \ldots)) \leq \frac{2}{3}4^{k-1}.$$

Therefore, as k tends to infinity, $\sigma^k(\mathbf{a})$ tends to $(\ldots, 1, 1, 1, \ldots)$. Since σ is invertible, there is a unique full solution $\gamma_{\mathbf{a}}$ given by $\gamma_{\mathbf{a}}(n) = \sigma^n(\mathbf{a})$. It is left to the reader to check that $\lim_{n\to-\infty} \sigma^n(\mathbf{a}) = (\ldots, 2, 2, 2, \ldots)$.

Definition 10.12 An element $x \in X$ is a *periodic point of period* p if there exists a positive integer p such that

$$f^p(x) = x.$$

p is the *minimal period* if $f^n(x) \neq x$ for $0 < n < p$. Observe that a *fixed point* is a periodic point with period 1. Given x_0, a periodic point of period p, its forward orbit $\{f^n(x_0) \mid n = 0, 1, \ldots, p-1\}$ is a *periodic orbit*.

Example 10.13 If $\mathbf{a} = (a_i)$ is a sequence satisfying $a_i = a_{i+p}$ for some positive p and all $i \in \mathbf{Z}$, then $\sigma^p(\mathbf{a}) = \mathbf{a}$, i.e. $\mathbf{a}$ is a periodic point of period p. Observe that

$$(\ldots, 1, 2, 1, 2, 1, 2, 1, 2, \ldots)$$

is a periodic point of period 4, but has minimal period 2.

Periodic points play an important role in symbolic dynamics and so we need to be able to express them using a simpler notation. Thus, if $\mathbf{a} = (a_i)$ is a periodic point of period p we will write

$$\mathbf{a} = (\overline{a_0, a_1, \ldots, a_{p-1}}).$$

Using this notation the periodic orbit associated to the periodic point of Example 10.13 can be written as

$$\left\{(\overline{1,2}), (\overline{2,1})\right\}$$

Just as periodic points generalize fixed points, we want to extend the notion of heteroclinic orbits to connecting orbits. However, to do this we need to generalize the notion of limits of orbits.

Definition 10.14 Let $x \in X$. The *omega limit set* of x under f is

$$\omega(x, f) := \bigcap_{m=0}^{\infty} \operatorname{cl}\left\{f^j(x) \mid j = m, m+1, \ldots\right\}$$

and the *alpha limit set* of a full solution γ_x of x under f is

$$\alpha(\gamma_x, f) := \bigcap_{m=0}^{\infty} \operatorname{cl}\left(\gamma_x((-\infty, -m))\right).$$

Proposition 10.15 *If X is a compact set and $f : X \to X$ is continuous, then for any $x \in X$, $\omega(x, f)$ is a non-empty compact invariant set.*

Proof. Given any integer $m \geq 0$, define

$$S_m := \operatorname{cl}\left\{f^j(x) \mid j = m, m+1, \ldots\right\}$$

Since S_m is a closed subset of the compact set X, it is compact and, by definition, it is nonempty. Furthermore, $S_{m+1} \subset S_m$ for each m. By Exercise 12.33,

$$\omega(x, f) = \bigcap_{m=0}^{\infty} S_m$$

is a non-empty compact set.

We still need to show that $\omega(x, f)$ is invariant. To do this it is sufficient to show that

$$f(\omega(x,f)) \subset \omega(x,f) \quad \text{and} \quad \omega(x,f) \subset f(\omega(x,f)).$$

To prove the first inclusion take $y \in \omega(x,f)$. Then there exists an increasing sequence of positive integers k_i such that $\lim_{i\to\infty} \operatorname{dist}(f^{k_i}(x), y) = 0$. Since f is continuous

$$\lim_{i\to\infty} \operatorname{dist}(f^{k_i+1}(x), f(y)) = 0.$$

Thus, $f(y) \in \omega(x,f)$ and $f(\omega(x,f)) \subset \omega(x,f)$.

To prove the other inclusion take $y \in \omega(x,f)$. Then $y = \lim_{i\to\infty} f^{k_i}(x)$ for some increasing sequence of integers k_i. Without loss of generality assume that $k_i \geq 1$ for all i and consider the sequence $\left\{f^{k_i-1}(x) \mid i = 1,2,3,\ldots\right\}$. Since X is compact, this contains a convergent subsequence $\left\{f^{k_{i_l}-1}(x) \mid l = 1,2,3,\ldots\right\}$. Let $z = \lim_{l\to\infty} f^{k_{i_l}-1}(x)$. Then $z \in \omega(x,f)$. Furthermore, because f is continuous

$$f(z) = \lim_{l\to\infty} f(f^{k_{i_l}-1}(x)) = \lim_{l\to\infty} f^{k_{i_l}}(x) = y.$$

Thus, $y \in f(\omega(x,f))$ and more generally $\omega(x,f) \subset f(\omega(x,f))$. □

Corollary 10.16 *If* $\mathbf{a} \in \Sigma_n$, *then* $\alpha(\mathbf{a},\sigma) \neq \emptyset$ *and* $\omega(\mathbf{a},\sigma) \neq \emptyset$.

Proof. By Proposition 10.6, Σ_n is compact. Since σ is invertible the same argument applies to alpha limit sets (see Exercise 10.2). □

Example 10.17 To see that alpha and omega limit sets are appropriate generalizations of the limits, consider $\mathbf{a} \in \Sigma_3$ from Example 10.11. Let us check that

$$\omega(\mathbf{a},\sigma) = (\bar{1}) \quad \text{and} \quad \alpha(\mathbf{a},\sigma) = (\bar{2}).$$

Since $\lim_{k\to\infty} \sigma^k(\mathbf{a}) = (\bar{1})$, $(\bar{1}) \in \operatorname{cl}\left\{\sigma^j(x) \mid j = m, m+1, \ldots\right\}$ for all m. Thus $(\bar{1}) \in \omega(\mathbf{a},\sigma)$. Let $\mathbf{b} = (b_i) \in \Sigma_n$ and assume $\mathbf{b} \neq (\bar{1})$. Then, there exists q such that $b_q \neq 1$. The inequality (10.1) allows us to conclude that $\operatorname{dist}(\mathbf{b}, \sigma^j(\mathbf{a})) \geq 4^{|q|}$ for every $j > |q|$. In particular,

$$\mathbf{b} \notin \operatorname{cl}\left\{\sigma^j(x) \mid j = |q|+1, |q|+2, \ldots\right\}$$

and hence $\mathbf{b} \notin \omega(\mathbf{a},\sigma)$.

A similar argument shows that $\alpha(\mathbf{a},\sigma) = (\bar{2})$.

Example 10.18 Alpha and omega limit sets can be quite complicated. Consider the point $\mathbf{a} = (a_i) \in \Sigma_2$ such that $\{a_i \mid i \geq 0\}$ defines the following sequence

1 2 11 12 21 111 112 121 211 212 1111 1112 1121 1211 1212 2111 2112 2121 ...

In words, the sequence is built of all possible blocks of 1 and 2 which do not contain two consecutive 2's. The sequence begins with blocks of length 1, followed by all blocks of length 2, then blocks of length 3, etc.

By Corollary 10.16, $\omega(\mathbf{a},\sigma)$ exists, but we can provide no simple description of all the orbits it contains.

Up to this point we have focused on orbits and invariant sets with a given structure and in this chapter we will provide theorems that allow one to prove the existence of particular orbits. However, for many nonlinear problems demonstrating the existence of particular orbits is extremely difficult and one is willing to settle for proving the existence of some invariant set within a prescribed region of phase space. This more modest goal leads to the following concept.

Definition 10.19 Let $f : X \to X$ be continuous and let $N \subset X$. The *maximal invariant set* in N under f is

$$\operatorname{Inv}(N, f) := \{x \in N \mid \text{ there exits a full solution } \gamma_x : \mathbf{Z} \to N\}.$$

As we indicated in the introduction, we will also use symbolic dynamics to understand the dynamics of other systems. However, to do this we will need a broader collection of examples than just the shift dynamics on n symbols. For this reason we need to introduce the notion of a *subshift of finite type*. Let $A = [a_{ij}]$ be a *transition matrix* on Σ_n, that is an $n \times n$ matrix with entries of the form $a_{ij} = 0, 1$. Define

$$\Sigma_A := \left\{\mathbf{a} \in \Sigma_n \mid a_{s_k s_{k+1}} = 1 \text{ for } k \in \mathbf{Z}\right\}.$$

It is easy to check that $\sigma(\Sigma_A) = \Sigma_A$. Therefore, we can define $\sigma_A : \Sigma_A \to \Sigma_A$ by restricting σ to Σ_A. The dynamical system $\sigma_A : \Sigma_A \to \Sigma_A$ is referred to as the *subshift of finite type for the transition matrix* A.

Example 10.20 Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a transition matrix on Σ_5. Let us consider the subshift of finite type $\sigma_A : \Sigma_A \to \Sigma_A$. There are several simple observations that can be made. The first is that there is only one fixed point

$$(\ldots, 1, 1, 1, 1, 1, \ldots).$$

The second is that there are no orbits that contain the symbols 3 or 5. This is easily seen from the fact that $a_{5j} = 0$, for all j and $a_{i3} = 0$, for all i. The third is that locally there are not many patterns of symbols that are permitted. To be more precise, given $\mathbf{s} \in \Sigma_A$

- If $s_0 = 1$, then $s_{\pm 1} \in \{1, 2\}$.
- If $s_0 = 2$, then $s_{\pm 1} \in \{1, 4\}$.
- If $s_0 = 4$, then $s_{-1} = s_1 = 2$.

On the other hand, even with all these restrictions σ_A still exhibits very rich dynamics. Observe, for example, that there are infinitely many periodic points including:

$$(\overline{2,1}) \quad (\overline{2,4}) \quad (\overline{2,1,2,4}).$$

From the previous example it should be clear that the form of the transition matrix A determines the how rich the dynamics of the corresponding subshift will be. With this in mind we make the following definition.

Definition 10.21 An $n \times n$ transition matrix A is *irreducible* if for every pair $1 \leq i, j \leq n$, there exists an integer $k = k(i,j)$ such that $\left(A^k\right)_{ij} \neq 0$.

Example 10.22 Let A be as in Example 10.20. Observe that A is not irreducible since $\left(A^k\right)_{5j} = 0$ for all $k \geq 0$ and all $1 \leq j \leq 5$. On the other hand if we consider the submatrix of A determined by the first, second and fourth columns and rows we obtain

$$A' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check that A' is irreducible since

$$(A')^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Observe that zero entries of $(A')^2$ are nonzero in A.

We leave the proof of the following proposition as an exercise.

Proposition 10.23 *Let A be an irreducible transition matrix. Then*

1. *The set of periodic orbits is dense in Σ_A.*
2. *There exists a dense orbit in Σ_A.*

Exercises

10.1 Prove that

$$\operatorname{dist}(\mathbf{a}, \mathbf{b}) := \sum_{j=-\infty}^{\infty} \frac{\delta(a_j, b_j)}{4^{|j|}}$$

defines a metric on Σ_n.

10.2 Suppose that $f : X \to X$ is a homeomorphism.

(a) Show that every $x \in X$ admits a unique full solution given by

$$\gamma_x(n) := f^n(x)\ ,\ n \in \mathbf{Z},$$

where $f^n := (f^{-1})^{|n|}$ for $n < 0$;

(b) Show that the solutions of f^{-1} are given by $\tilde{\gamma}_x(n) = \gamma_x(-n)$ where γ_x is a solution of f;
(c) For $N \subset X$, show that $\operatorname{Inv}(N, f^{-1}) = \operatorname{Inv}(N, f)$;
(d) By the uniqueness of full solutions established in (a), it makes sense to write $\alpha(x, f)$ rather than $\alpha(\gamma_x, f)$. Show that

$$\alpha(x, f^{-1}) = \omega(x, f) \text{ and } \omega(x, f^{-1}) = \alpha(x, f).$$

10.3 Construct an example of a continuous map in which some elements have multiple full solutions.

10.4 Let S be an invariant set under the continuous map $f : X \to X$. Prove that if $x \in S$, then there exists a full solution $\gamma_x : \mathbf{Z} \to S$.

10.5 List all periodic orbits of period 3, 4, and 5 for $\sigma : \Sigma_2 \to \Sigma_2$, the full shift on two symbols. How many periodic orbits of period n are there?

10.6 Prove Proposition 10.23.

10.2 Differential Equations and Flows

Historically the theory of dynamical systems has its roots in the qualitative theory of ordinary differential equations. Thus, we now turn our attention to this topic and briefly review some standard results whose proofs can be found e.g. in [24, 26].

Let $f : \mathbf{R}^d \to \mathbf{R}^d$. Consider an open interval $I \subset \mathbf{R}$. A differentiable function

$$\begin{aligned} x : I &\to \mathbf{R}^d \\ t &\mapsto x(t) \end{aligned}$$

is a *solution* to the ordinary differential equation

$$\dot{x} = f(x)$$

where the dot stands for the derivative with respect to t, if $\frac{dx}{dt}(t) = f(x(t))$ for all $t \in I$. Usually the t variable in this context is referred to as *time*. An initial value problem is formulated as follows: given the initial time $t = 0$ and an initial condition $x_0 \in \mathbf{R}^d$ find a solution $x : I \to \mathbf{R}^d$ with the property that $x(0) = x_0$. The following theorem gives conditions under which this problem has a unique solution.

Theorem 10.24 *(Existence and Uniqueness Theorem) If $f \in C^1(\mathbf{R}^d, \mathbf{R}^d)$, then for any $x_0 \in \mathbf{R}^d$, there exists an open interval I (possibly unbounded) containing* 0 *and a unique solution $\varphi(\cdot, x_0) : I \to \mathbf{R}^d$ of $\dot{x} = f(x)$ with the property that $\varphi(0, x_0) = x$.*

To simplify the discussion, we shall usually consider functions f such that for each $x \in \mathbf{R}^d$ the solution is defined for all $t \in \mathbf{R}$. Observe that this implies that $\varphi : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$. By arguments using the chain rule for derivation one can prove that $\varphi(t+s,x) = \varphi(t, \varphi(s,x))$ (see [26]). Furthermore, $\varphi(0,x) = x$. The fact that for fixed x, $\varphi(\cdot, x)$ is a solution implies that φ is differentiable in t. That φ is a continuous function of both variables follows from a classical result known as continuity with respect to initial conditions. ([24, Theorem 3.4]). Abstracting this leads to the following definitions.

Definition 10.25 Let X be a topological space. A continuous function $\varphi : \mathbf{R} \times X \to X$ is a *flow* if:

1. $\varphi(0,x) = x$ for all $x \in X$, and
2. $\varphi(t+s,x) = \varphi(t, \varphi(s,x))$ for all $t, s \in \mathbf{R}$ and $x \in X$.

Definition 10.26 The *orbit* of the point $x \in X$ under the flow φ is

$$\varphi(\mathbf{R}, x) := \{\varphi(t,x) \mid t \in \mathbf{R}\}.$$

Observe that an orbit in a flow is analogous to the solution curve of a differential equation.

Since the goal of dynamical systems is to understand the existence and structure of orbits, we are once again led to the following concepts.

Definition 10.27 A subset $S \subset \mathbf{R}^d$ is *invariant* under the flow φ, if for every element $x \in S$, its orbit $\varphi(\mathbf{R}, x) \subset S$, or equivalently $\varphi(t,S) = S$ for every $t \in \mathbf{R}$.

Definition 10.28 The *maximal invariant set* in $N \subset X$ under the flow φ is

$$\operatorname{Inv}(N, \varphi) := \{x \in N \mid \varphi(\mathbf{R}, x) \subset N\}.$$

Definition 10.29 Let $x \in X$. The *alpha* and *omega limit set* of x under φ are

$$\alpha(x,\varphi) := \bigcap_{t \leq 0} \operatorname{cl}\,(\varphi((-\infty,t),x)) \quad \text{and} \quad \omega(x,\varphi) := \bigcap_{t \geq 0} \operatorname{cl}\,(\varphi([t,\infty),x)).$$

A proof similar to the one used to demonstrate Proposition 10.15 leads to the following result.

Proposition 10.30 *If X is a compact set, then for any $x \in X$ the sets $\alpha(x,\varphi)$ and $\omega(x,\varphi)$ are non-empty compact and invariant.*

We can make definitions concerning the types of orbits of flows that are similar to those for maps.

Definition 10.31 An element $x \in X$ is an *equilibrium point* of φ if $\varphi(t,x) = x$ for all $t \in \mathbf{R}$. It is a *periodic point* of period $\tau > 0$ if $\varphi(\tau, x) = x$.

We can go a step further and observe that there is a very strong relationship between flows and maps. In particular, given a flow $\varphi : \mathbf{R} \times X \to X$ we can always define the *time-*τ map $\varphi_\tau : X \to X$ by fixing a particular time τ and letting

$$\varphi_\tau(x) := \varphi(\tau, x) \tag{10.3}$$

This relationship demonstrates that the problem of finding periodic points for flows is closely related to finding fixed points of maps. Clearly, x is a periodic point for φ with period τ if and only if x is a fixed point of φ_τ.

The reader should also recognize that time-τ maps represent a very special class of maps.

Proposition 10.32 *Let $\varphi : \mathbf{R} \times X \to X$ be a flow. Then φ_τ is homotopic to the identity map, i.e.*

$$\varphi_\tau \sim \mathrm{id}_X$$

Proof. The flow itself provides the homotopy. To be more precise define $h : X \times [0,1] \to X$ by $h(x,s) = \varphi(s\tau, x)$. Since φ is continuous h is continuous. Furthermore, $h(0,x) = \varphi(0,x) = x$ and $h(1,x) = \varphi(\tau,x) = \varphi_\tau(x)$. □

10.3 Ważewski Principle

In the previous two sections we introduced basic concepts from dynamical systems. We now turn to the question of using topology to prove the existence of an invariant set within a specified region. In the case of flows there is an elementary but deep method called the Ważewski principle.

Let X be a topological space and let $\varphi : \mathbf{R} \times X \to X$ be a flow. Consider $W \subset X$. The set of points in W which eventually leave in forward time is

$$W^0 = \{x \in W \mid \text{ there exists } t > 0, \varphi(t,x) \notin W\}.$$

The set of points which immediately leave W is denoted by

$$W^- = \{x \in W \mid \text{ for all } t > 0, \varphi([0,t),x) \not\subset W\}.$$

Clearly, $W^- \subset W^0 \subset W$. Observe that it is possible that $W^0 = \emptyset$, in which case, $W^- = \emptyset$.

Definition 10.33 W is a *Ważewski set* if the following conditions are satisfied.

1. $x \in W$ and $\varphi([0,t],x) \subset \mathrm{cl}(W)$ implies that $\varphi([0,t],x) \subset W$.
2. W^- is closed relative to W^0.

Theorem 10.34 *Let W be a Ważewski set. Then W^- is a strong deformation retract of W^0 and W^0 is open relative to W.*

Proof. Observe that if $W^0 = \emptyset$, then the result is trivially true. Therefore assume that $W^0 \neq \emptyset$. To construct the strong deformation retraction

$$h : W^0 \times [0,1] \to W^0$$

of W^0 onto W^-, we first define $\tau : W^0 \to [0,\infty)$ by

$$\tau(x) = \sup\{t \geq 0 \mid \varphi([0,t],x) \subset W\}.$$

By the definition of W^0, $\tau(x)$ is finite. Continuity of the flow implies that $\varphi([0,\tau(x)],x) \subset \mathrm{cl}\,(W)$.

Since W is a Ważewski set, $\varphi(\tau(x),x) \in W$, and in fact the definition of τ implies that $\varphi(\tau(x),x) \in W^-$. Observe that $\tau(x) = 0$ if and only if $x \in W^-$.

Assume for the moment that τ is continuous and define h by

$$h(x,\sigma) = \varphi((\sigma\tau(x)),x).$$

Obviously,

$$h(x,0) = \varphi(0,x) = x$$
$$h(x,1) = \varphi(\tau(x),x) \in W^-$$

and for $y \in W^-$

$$h(y,\sigma) = \varphi(\sigma\tau(y),y) = \varphi(0,y) = y$$

Therefore r is a strong deformation retraction of W^0 onto W^-.

The proof that τ is continuous will be done in two steps. The first is to show that τ is upper semicontinuous (see Definition 12.36 and Proposition 12.37). Let $x \in W^0$ and $\epsilon > 0$, then $\varphi([\tau(x),\tau(x)+\epsilon],x) \not\subset W$. By the first property of Definition 10.33 there exists $t_0 \in [\tau(x),\tau(x)+\epsilon]$ such that $\varphi(t_0,x) \notin \mathrm{cl}(W)$. Thus we can choose V to be a neighborhood of $\varphi(t_0,x)$ such that $V \cap \mathrm{cl}(W) = \emptyset$. Now let U be a neighborhood of x such that $\varphi(t_0,U) \subset V$. Then for $y \in U \cap W$, $\varphi(t_0,y) \notin W$. Note that this proves that W^0 is open relative to W and that $\tau(y) < \tau(x) + \epsilon$, i.e. τ is upper semi-continuous.

Obviously τ is lower semi-continuous at W^-. To prove lower semi-continuity of τ at the other points, let $x \in W^0 \setminus W^-$ and let $0 < \epsilon < \tau(x)$. Then $\varphi([0,\tau(x)-\epsilon],x) \subset W^0$. Since $\epsilon > 0$, we see that $\varphi([0,\tau(x)-\epsilon],x) \cap W^- = \emptyset$. Hence for every $s \in [0,\tau(x)-\epsilon]$ there exists a neighborhood U_s of $\varphi(s,x)$ such that $U_s \cap W^- = \emptyset$. Since φ is continuous, by Proposition 12.66 $\varphi([0,\tau(x)-\epsilon],x)$ is compact. Since $\{U_s\}$ covers $\varphi([0,\tau(x)-\epsilon],x)$ Theorem 12.69 implies that there is a finite subcovering $\{U_{s_i} \mid i = 1,\dots n\}$ of $\varphi([0,\tau(x)-\epsilon],x)$. Let $U = \cup_{i=1}^n U_{s_i}$, then U is open which implies there exists a neighborhood V of x such that $\varphi([0,\tau(x)-\epsilon],V) \subset U$. Now $U \cap W^- = \emptyset$ which implies that for all $y \in V$ we have $\varphi([0,\tau(x)-\epsilon],y) \cap W^- = \emptyset$. Therefore, $\tau(y) \geq \tau(x) - \epsilon$ which shows that τ is lower semicontinuous, and hence, continuous. □

Corollary 10.35 [Ważewski Principle] *If W is a Ważewski set and W^- is not a strong deformation retract of W then $W \backslash W^0 \neq \emptyset$, i.e., there exist solutions which stay in W for all positive time.*

Proof. Suppose that, on the contrary, all solutions eventually leave W. Then $W^0 = W$ so we get a contradiction with Theorem 10.34. □

Corollary 10.36 *If W is a compact Ważewski set and W^- is not a strong deformation retract of W, then* $\operatorname{Inv}(W, \varphi) \neq \emptyset$.

Proof. By Corollary 10.35, there exists a point $x \in W$ such that $\varphi([0, \infty), x) \subset W$. By Proposition 10.30, $\omega(x, \varphi)$ is a non-empty compact invariant subset of W. □

Hopefully, the reader is impressed by the elegant relation between topology and dynamics expressed in the Ważewski principle. However, this is a book on algebraic topology and so it is reasonable to wonder: how does homology fit into the picture?

Proposition 10.37 *Let W be a Ważewski set. Assume W and W^- are cubical. If $H_*(W, W^-) \neq 0$, then* $\operatorname{Inv}(W, \varphi) \neq \emptyset$.

Proof. using exact sequence of a pair □

Let us now consider two examples. The first shows the power of Ważewski's principle and the second, an inherent weakness.

Example 10.38 Consider the differential equation

$$\dot{x} = Ax \tag{10.4}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The reader can check that

$$\varphi\left(t, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} e^t x_1 \\ e^{-t} x_2 \end{bmatrix}$$

is a solution to (10.4). Observe that the only bounded invariant set is the equilibrium solution; every other orbit is unbounded in either forward or backward time.

Now consider the following differential equation

$$\dot{x} = Ax + f(x) \tag{10.5}$$

where we do not know f explicitly, but assume that it generates a flow ψ. Instead, all the information we are given is that $\|f(x)\| < \kappa$ for all $x \in \mathbf{R}^2$. We would like to know if (10.5) has a bounded solution. Let $m = \operatorname{ceil}(\kappa)$ and let $W = [-m, m]^2 \subset \mathbf{R}^2$ (see Figure 10.1). We claim that W is a Ważewski set and

$$W^- = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in W \,\middle|\, |x_1| = m \right\}. \tag{10.6}$$

Observe that if (10.6) is correct, then W is a Ważewski set since both W and W^- are closed.

Clearly W^- is a subset of the boundary of W (otherwise the point could not leave W immediately in forward time). We shall verify (10.6) by evaluating vector field $Ax + f(x)$ at points on the boundary of W. Let

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}.$$

The assumption that $\|f(x)\| < \kappa$ implies that $\|f_i(x)\| < \kappa$ for $i = 1, 2$. Consider

$$x = \begin{bmatrix} m \\ x_2 \end{bmatrix}$$

where $|x_2| \leq m$. Observe that at x,

$$\dot{x}_1 = m + f_1(x) > m - \kappa \geq 0.$$

Thus, the second coordinate of the solution through x increases as time increases, therefore it leaves W immediately in forward time, i.e. $x \in W^-$. We leave it to the reader to check the other cases and to check that

$$H_1(W, W^-) \cong \mathbf{Z}.$$

Therefore, by Proposition 10.37, $\text{Inv}\,(W, \psi) \neq \emptyset$.

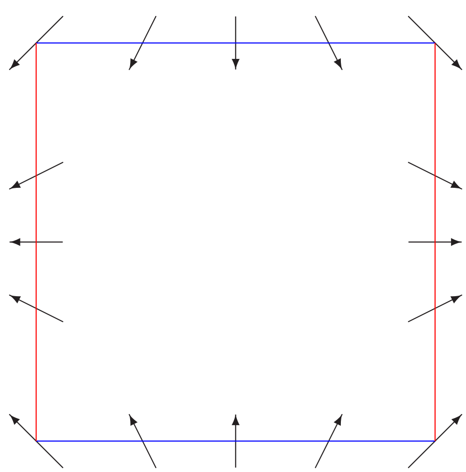

Fig. 10.1. A Ważewski set W with W^- indicated in red.

Example 10.39 Consider the following one-parameter family of scalar ordinary differential equations

$$\dot{x} = x^2 + \lambda^2$$

where $\lambda \in [-1, 1]$. Let $W_\lambda = [-1, 0]$. Then W_λ is a Ważewski set. Notice now that $W^{0-} = \emptyset$, and hence, Ważewski's principle detects the existence of a non-trivial invariant set, i.e., the equilibrium $\{0\}$, for $\lambda = 0$ since W_0^- is not a strong deformation retract of W_0. However, for all other values of λ the invariant set is trivial. This is an important point. The use of Ważewski's principle to prove the existence of an invariant set at one parameter value does not allow one to conclude the existence of an invariant set at any other parameter value. In other words, Ważewski's principle is not robust with respect to perturbations.

10.4 Fixed Point Theorems

While the Ważewski principle and index pairs provide a method to prove the existence of a nonempty invariant set, it gives no information concerning the structure of the invariant set. With this in mind, we change topics slightly and consider some classical fixed point theorems which are outgrowths of homotopy invariance introduced in Section 6.5. As will become clear these theorems allow one conclude the existence of an equilibrium or fixed point by means of an algebraic topological computation.

We start from presenting classical theorems relating homotopical properties of the sphere to fixed points of a continuous map on the corresponding disk i.e. a closed ball. We discuss unit spheres and balls with respect to the supremum norm because they are cubical sets but all statements extend to spheres and balls with respect to any norm in $\mathbf{R}^d$ once the notion of homology is generalized to topological polyhedra in Chapter 11.

10.4.1 Fixed Points in The Unit Ball

Recall that $\bar{B}_0^{d+1} := [-1, 1]^{d+1}$, the unit ball with radius 1 and $S_0^d = \operatorname{bd} \bar{B}_0^{d+1}$ is the corresponding unitary sphere. Let $f : \bar{B}_0^{d+1} \to \bar{B}_0^{d+1}$ be a continuous map. If one wishes to remain in the setting of flows, consider $\varphi : \mathbf{R} \times \bar{B}_0^{d+1} \to \bar{B}_0^{d+1}$ and let $f = \varphi_\tau$ for some $\tau > 0$.

The following theorem due to K. Borsuk (1931, see [15]) brings together three famous theorems: the Non-contractibility theorem, the Non-retraction theorem, and the Brouwer Fixed Point Theorem (1912).

Theorem 10.40 *Let $d \geq 0$. The following statements are equivalent:*

(a) S_0^d is not contractible.
(b) Every continuous map $f : \bar{B}_0^{d+1} \to \bar{B}_0^{d+1}$ has a fixed point.
(c) There is no retraction from $\bar{B}_0^{d+1}$ onto S_0^d.

Proof. All implications will be proved by contradiction.

We first show that (a) implies (b). Suppose that $f(x) \neq x$ for all $x \in \bar{B}_0^{d+1}$. Then $y - tf(y) \neq 0$ for all $y \in S_0^d$ and all $t \in [0,1]$. Indeed, if $t = 1$ that is immediate from what we just supposed, and if $0 \leq t < 1$, it follows from the inequality

$$\|tf(y)\|_0 < \|f(y)\|_0 \leq 1 = \|y\|_0.$$

Let $n : \mathbf{R}^{d+1} \setminus \{0\} \to S_0^d$ be given by $n(x) = \frac{x}{\|x\|_0}$. The map $h_1 : S_0^n \times [0,1] \to S_0^n$ given by

$$h_1(y,t) = n(y - tf(y))$$

is a homotopy from $\mathrm{id}_{S_0^d}$ to the map $g(y) = n(y - f(y))$ and the map h_0 given by

$$h_0(y,t) = n((1-t)y - f((1-t)y))$$

is a homotopy from g to the constant map $k(y) = y_0 := n(-f(0))$. By the transitivity property (see Exercise 6.14), $\mathrm{id}_{S_0^d} \sim k$, a contradiction.

We next show that (b) implies (c). Suppose that there is a retraction $r : \bar{B}_0^{d+1} \to S_0^d$. Then $f : \bar{B}_0^{d+1} \to \bar{B}_0^{d+1}$, $f(x) = -r(x)$ is fixed point free.

We finally show that (c) implies (a). Suppose that there is a homotopy $h : S_0^n \times [0,1] \to S_0^n$ joining $\mathrm{id}_{S_0^n}$ at $t = 0$ to $y_0 \in S_0^n$ at $t = 1$. Then the formula

$$r(x) := \begin{cases} y_0 & \text{if } \|x\| \leq \frac{1}{2} \\ h(n(x), 2(1 - \|x\|_0)) & \text{if } \|x\| \geq \frac{1}{2} \end{cases}$$

defines a retraction r of $\bar{B}_0^{d+1}$ onto S_0^d. □

Observe that this theorem merely proves the equivalence of these statements, but does not address their truthfulness. This is resolved in the following corollary.

Corollary 10.41 *All three statements of Theorem 10.40 are true.*

Proof. By Exercise 9.10 for $d \geq 1$

$$H_k(S_0^d) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0, d \\ 0 & \text{otherwise,} \end{cases}$$

while for $d = 0$

$$H_k(S_0^d) \cong \begin{cases} \mathbf{Z}^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In either case, it is clear that S_0^d is not acyclic and hence by Corollary 6.68 is not contractible to a point. Thus, condition (a) of Theorem 10.40 holds and therefore the remaining two do to. □

10.4.2 Lefschetz Fixed Point Theorem

The Brouwer fixed point theorem of the previous section has the advantage that it easy to state and has an elementary proof in the sense that it only requires knowledge of continuous functions. However, it is very limited since it only applies to disks in $\mathbf{R}^d$. The Lefschetz fixed point theorem, which will be presented in this section, is one of the most important results in algebraic topology. In particular, it allows one to conclude the existence of a fixed point of a continuous map f based on the associated homology map f_*. Of course, one must pay a price for such a gem; the proof is no longer elementary - it involves a delicate mixture of algebra and topology. We begin, therefore, with a few necessary algebraic preliminaries.

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *trace* of A is defined to be the sum of the diagonal entries, that is,

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

If A and B are $n \times n$ matrices, then

$$\operatorname{tr} AB = \sum_{i,j=1}^{n} a_{ij} b_{ji} = \sum_{i,j=1}^{n} b_{ji} a_{ij} = \operatorname{tr} BA.$$

Let G be a finitely generated free abelian group and let $\phi : G \to G$ be a group homomorphism. Since G is free abelian, it has a basis and for a particular choice of basis ϕ can be written as a matrix A. So in this case define

$$\operatorname{tr} \phi = \operatorname{tr} A.$$

To check that this is a well defined concept, let $\{b_, \ldots, b_n\}$ be a different basis for G.

Let $B : G \to G$ be an isomorphism corresponding to a change of basis. By Corollary 3.6 in the new basis the matrix representation of ϕ is given by $B^{-1}AB$. Observe that,

$$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(B^{-1}(AB)) = \operatorname{tr}((AB)B^{-1}) = \operatorname{tr} A.$$

Given a homomorphism $f : G \to G$ where G is a finitely generated abelian group, we define the $\operatorname{tr} f$ as the $\operatorname{tr} \tilde{f}$ where $\tilde{f} : G/T(G) \to G/T(G)$ is the induced homomorphism of free abelian groups (see Corollary 3.59).

Definition 10.42 Let X be a cubical set and let $f : X \to X$ be a continuous map. The *Lefschetz number* of f is

$$L(f) := \sum_{k} (-1)^k \operatorname{tr} f_k.$$

Theorem 10.43 Lefschetz Fixed Point Theorem *Let X be a cubical set and let $f : X \to X$ be a continuous map. If $L(f) \neq 0$, then f has a fixed point.*

This theorem is an amazing example of how closely the algebra is tied to the topology. To prove it we need to understand how to relate the topology in the form of the map on the chain complexes to the algebra in the form of the induced homology maps on the free part of the homology groups.

We begin with a technical lemma.

Lemma 10.44 *Consider a finitely generated free abelian group G and a subgroup H such that G/H is free abelian. Let $\phi : G \to G$ be a group homomorphism such that $\phi(H) \subset H$. Then,*

$$\operatorname{tr}\phi = \operatorname{tr}\phi' + \operatorname{tr}\phi\,|_H\ .$$

where $\phi' : G/H \to G/H$ is the induced homorphism

Proof. By Lemma 13.68, the fact that G is a finitely generated free abelian group and that H is a subgroup implies that H is also a finitely generated free abelian group. Let $\{v_1, \ldots, v_k\}$ be a basis for H and let $\{u_1 + H, \ldots, u_n + H\}$ be a basis for G/H. Since ϕ and ϕ' are group homomorphisms there exists integers α_{ij} and β_{ij} such that

$$\phi'(u_j + H) = \sum_{i=1}^{n} \beta_{ij}(u_i + H)$$

$$\phi\,|_H\,(v_i) = \sum_{i=1}^{k} \alpha_{ij} v_i.$$

where the latter equation follows from the fact that $\phi(H) \subset H$. Observe that using these bases the matrix representations of ϕ and ϕ' are given by $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ respectively.

By Exercise 13.18 $G \cong G/H \oplus H$, thus $\{u_1, \ldots, u_n, v_1, \ldots, v_k\}$ is a basis for G. Furthermore,

$$\phi(u_j) = \sum_{i=1}^{n} \beta_{ij} u_i + h_j \quad \text{for some } h_j \in H$$

$$\phi(v_j) = \sum_{i=1}^{k} \alpha_{ij} v_i$$

This means that using this basis the matrix representation of ϕ has the form

$$\begin{bmatrix} B & * \\ 0 & A \end{bmatrix}.$$

Clearly, $tr\phi = \operatorname{tr}\phi' + \operatorname{tr}\phi\,|_H$. □

The following theorem plays a crucial role in that it allows us to relate the trace of the homology map to the chain map which induced it.

Theorem 10.45 (Hopf trace theorem) *Let $\{C_k(X), \partial_k\}$ be a finitely generated free chain complex and $\varphi : C(X) \to C(X)$ a chain map. Let $H_k(X)$ denote the corresponding homology groups with torsion subgroups $T_k(X)$. Let $\phi_k : H_k(X)/T_k(X) \to H_k(X)/T_k(X)$ be the induced homomorphism. Then*

$$\sum_k (-1)^k \operatorname{tr} \varphi_k = \sum_k (-1)^k \operatorname{tr} \phi_k.$$

Proof. We will use the notation from Section 9.5 where $W_k(X)$ denotes the weak boundaries. Recall that

$$B_k(X) \subset W_k(X) \subset Z_k(X) \subset C_k(X).$$

Furthermore, since φ is a chain map, each of these subgroups is invariant under φ_k, i.e. $\varphi_k(B_k(X)) \subset B_k(X)$, $\varphi_k(W_k(X)) \subset W_k(X)$, etc. By Lemma 10.44 φ_k induces maps

$$\begin{aligned} \varphi_k \mid_{W_k(X)} &: W_k(X) \to W_k(X), \\ \varphi'_k &: Z_k(X)/W_k(X) \to Z_k(X)/W_k(X) \\ \varphi''_k &: C_k(X)/Z_k(X) \to C_k(X)/Z_k(X). \end{aligned}$$

From (9.17) we have that $Z_k(X)/W_k(X)$ is free abelian for each k. Using Exercise 13.15 it follows that $C_k(X)/Z_k(X)$ is free abelian for each k. Therefore, applying Lemma 10.44 twice gives

$$\operatorname{tr} \varphi_k = \operatorname{tr} \varphi''_k + \operatorname{tr} \varphi'_k + \operatorname{tr} \varphi_k \mid_{W_k(X)} . \tag{10.7}$$

Again, from (9.17) $Z_k(X)/W_k(X) \cong H_k(X)/T_k(X)$ and furthermore under this isomorphism, φ_k becomes ϕ_k. Therefore, (10.7) becomes

$$\operatorname{tr} \varphi_k = \operatorname{tr} \varphi''_k + \operatorname{tr} \phi_k + \operatorname{tr} \varphi_k \mid_{W_k(X)} . \tag{10.8}$$

Similarly, $C_k(X)/Z_k(X)$ is isomorphic to $B_{k-1}(X)$ and under this isomorphism φ''_k becomes $\varphi_{k-1} \mid_{B_{k-1}(X)}$. Hence, (10.8) can be written as

$$\operatorname{tr} \varphi_k = \operatorname{tr} \varphi_{k-1} \mid_{B_{k-1}(X)} + \operatorname{tr} \phi_k + \operatorname{tr} \varphi_k \mid_{W_k(X)} . \tag{10.9}$$

We will now show that $\operatorname{tr} \varphi_k \mid_{W_k(X)} = \operatorname{tr} \varphi_k \mid_{B_k(X)}$. As was done explicitly in Section 9.5 there exists a basis $\{u_1, \ldots, u_l\}$ for $W_k(X)$ and integers $\alpha_1, \ldots, \alpha_l$, such that $\{\alpha_1 u_1, \ldots, \alpha_l u_l\}$ is a basis for $B_k(X)$.

Observe that

$$\varphi_k \mid_{W_k(X)} (u_j) = \sum_{i=1}^{l} a_{ij} u_i \tag{10.10}$$

and

$$\varphi_k \mid_{B_k(X)} (\alpha_j u_j) = \sum_{i=1}^{l} b_{ij} \alpha_i u_i \tag{10.11}$$

for appropriate constants a_{ij} and b_{ij}. Both these maps are just restrictions of φ_k to the appropriate subspaces. So multiplying (10.10) by α_j must give rise to (10.11) and hence $\alpha_j a_{ij} = b_{ij}\alpha_i$ and in particular $\alpha_i a_{ii} = b_{ii}\alpha_i$. Therefore, $\operatorname{tr}\varphi_k \mid_{W_k(X)} = \operatorname{tr}\varphi_k \mid_{B_k(X)}$. Applying this to (10.9) give

$$\operatorname{tr}\varphi_k = \operatorname{tr}\varphi_{k-1} \mid_{B_{k-1}(X)} + \operatorname{tr}\phi_k + \operatorname{tr}\varphi_k \mid_{B_k(X)} . \tag{10.12}$$

The proof is finished by multiplying (10.12) by $(-1)^k$ and summing. □

The Hopf trace formula is the key step in the proof of the Lefschetz fixed point theorem. However, before beginning the proof let us discuss the basic argument that will be used. Observe that an equivalent statement to the Lefschetz fixed point theorem is the following: *if f has no fixed points, then* $L(f) = 0$. This is what we will prove. The Hopf trace formula provides us with a means of relating a chain map $\varphi : C(X) \to C(X)$ for f with $L(f)$. In particular, if we could show that $\operatorname{tr}\varphi = 0$, then it would be clear that $L(f) = 0$. Of course, the easiest way to check that $\operatorname{tr}\varphi = 0$ is for all the diagonal entries of φ to be zero. However, the diagonal entries of φ indicate how the duals of elementary cubes are mapped to themselves. If f has no fixed points then the image of a small cube will not intersect itself and so the diagonal entries are zero. With this argument in mind we turn to the proof, which as is often the case in mathematics, is presented in the reverse order.

Proof of Lefschetz Fixed Point Theorem. We argue by contradiction. Assume f has no fixed points. We want to show that $L(f) = 0$. Let

$$\epsilon := \inf_{x \in X} ||x - f(x)||$$

where $\|\cdot\| = \|\cdot\|_0$ as defined by (12.3). Observe that $\epsilon > 0$. Indeed, since cubical sets are compact and the function $x \mapsto ||x - f(x)||$ is continuous, it has a minimum at some $x_0 \in X$ (see Theorem 12.67). But we are assuming that f has no fixed points so $\epsilon = ||x_0 - f(x_0)|| > 0$. Take any integer $m > 2/\epsilon$ and a scaling vector α with $\alpha_i = m$ for all i. We will study the map

$$g := \Lambda^\alpha_X \circ f \circ \Omega^\alpha_X : X^\alpha \to X^\alpha.$$

Note that for any $y \in X^\alpha$

$$||y - g(y)|| > 2. \tag{10.13}$$

Indeed, let $x = \Omega^\alpha(y)$. Then

$$\begin{aligned} ||y - g(y)|| &= ||\Lambda^\alpha(\Omega^\alpha(y) - f(\Omega^\alpha(y)))|| \\ &= ||\Lambda^\alpha(x - f(x))|| = m||x - f(x)|| \\ &> \frac{2}{\epsilon} \cdot \epsilon = 2. \end{aligned}$$

We next prove a combinatorial version of the inequality $x \neq f(x)$ for g, that is,

$$Q \cap \operatorname{ch}(g(Q)) = \emptyset. \tag{10.14}$$

Suppose the contrary and let $x \in Q \cap \operatorname{ch}(g(Q))$. By the definition of closed hull, $\operatorname{dist}(x, g(Q)) \leq 1$ so there exists $y \in Q$ such that $||x - g(y)|| \leq 1$. Since $x, y \in Q$ it follows that $||x - y|| \leq 1$. Therefore

$$||y - g(y)|| \leq ||y - x|| + ||x - g(y)|| < 2,$$

which contradicts (10.13).

Let now Λ^β be a rescaling of X^α such that

$$M_{g^\beta} : X^{\alpha\beta} \to X^\alpha$$

is acyclic valued. Let θ be a chain selector of $M_{\Lambda^\beta} : X^\alpha \rightrightarrows X^{\alpha\beta}$ and ψ be a chain selector of M_{g^β}. Then $g_* := \psi_* \circ \theta_* = (\psi \circ \theta)_*$. We next show that

$$Q \cap |\psi \circ \theta(\widehat{Q})| = \emptyset \tag{10.15}$$

for all $Q \in \mathcal{K}(X^\alpha)$. Indeed, suppose the contrary and let $x \in Q \cap |\psi \circ \theta(\widehat{Q})|$. Since

$$|\psi \circ \theta(\widehat{Q})| \subset \bigcup \{M_{g^\beta}(\overset{\circ}{P}) \mid P \in \mathcal{K}(|\theta(\widehat{Q})|)\},$$

there exists an elementary cube $P \subset |\theta(\widehat{Q})|$ such that $x \in M_{g^\beta}(\overset{\circ}{P})$. We have

$$\begin{aligned} |\theta(\widehat{Q})| \subset M_{\Lambda^\beta}(\overset{\circ}{Q}) &= \operatorname{ch}(\Lambda^\beta(\operatorname{ch}(\overset{\circ}{Q}))) \\ &= \operatorname{ch}(\Lambda^\beta(Q)) = \Lambda^\beta(Q), \end{aligned}$$

so $P \subset \Lambda^\beta(Q)$. Hence we get

$$\begin{aligned} x \in M_{g^\beta}(\overset{\circ}{P}) &= \operatorname{ch}(g^\beta(\operatorname{ch}(\overset{\circ}{P}))) = \operatorname{ch}(g \circ \Omega^\beta(P)) \\ &\subset \operatorname{ch}(g \circ \Omega^\beta(\Lambda^\beta(Q))) = \operatorname{ch}(g(Q)), \end{aligned}$$

so $x \in \operatorname{ch}(g(Q))$ which contradicts (10.14). Equation (10.15) implies that the diagonal entries in the matrix of the chain map $\psi \circ \theta$ with respect to the basis $\widehat{\mathcal{K}}(X^\alpha)$ are all zero, hence $\operatorname{tr} \psi \circ \theta = 0$. By the Hopf trace formula, $L(g_*) = 0$. Finally, by Proposition 6.51,

$$L(f_*) = L(g_*) = 0.$$

□

Theorem 10.46 *Let X be an acyclic cubical set. Let $f : X \to X$ be continuous. Then, f has a fixed point.*

Proof. Since X is acyclic, the only nonzero homology group is $H_0(X) \cong \mathbf{Z}$. But, by Proposition 6.59, $f_* : H_0(X) \to H_0(X)$ is the identity map. Therefore, $L(f) = 1$. □

Corollary 10.47 *Let $\varphi : \mathbf{R} \times X \to X$ be a flow. If X is an acyclic cubical set, then φ has an equilibrium.*

Proof. Observe that for each $\tau > 0$, $\varphi_\tau : X \to X$ has a fixed point. In particular, let t_n be a sequence of positive times such that $\lim_{n\to\infty} t_n = 0$. For each t_n there exists $x_n \in X$ such that $\varphi_{t_n}(x_n) = x_n$. Since, X is compact, $\{x_n\}$ contains a convergent subsequence $\{x_{n_m}\}$. Let $y = \lim_{m\to\infty} x_{n_m}$. Finally, let s_m be the minimal period of x_{n_m}.

We prove that y is an equilibrium, i.e. that $\varphi(t, y) = y$ for any $t \in \mathbf{R}$. Assume not. Then, there exists $\tau > 0$ such that $\varphi(\tau, y) = z \neq y$. Since $z \neq y$, there exists $\epsilon > 0$ such that $\|z - y\| > \epsilon$. By continuity

$$\lim_{m\to\infty} \varphi(\tau, x_{n_m}) = z.$$

Now consider the set $\varphi([0, s_m), x_{n_m})$. Since x_{n_m} is periodic with period s_m, $\varphi(\mathbf{R}, x_{n_m}) = \varphi([0, s_m), x_{n_m})$. Since $\lim_{m\to\infty} s_m = 0$ and φ is continuous,

$$\lim_{m\to\infty} \operatorname{diam}(\varphi(\mathbf{R}, x_{n_m})) = 0$$

But this contradicts the fact that $\|z - y\| > \epsilon$. □

Both Theorem 10.46 and Corollary 10.47 follow from the fact that the Lefschetz number of the identity map on an acyclic space is non-zero because the only nontrivial homology H_0. The following concept generalizes this idea.

Definition 10.48 Let X be a cubical set. The *Euler number* of X is

$$E(X) := \sum_{i=0}^{\infty} (-1)^i \beta_i(X)$$

where $\beta_i(X)$ is the Betti number of $H_i(X)$.

The proof of the following result is almost identical to that of Corollary 10.47.

Theorem 10.49 *Let $\varphi : \mathbf{R} \times X \to X$ be a flow on a cubical set. If $E(X) \neq 0$, then φ has an equilibrium.*

Exercises

10.7 Let X be a cubical set and let $\varphi : \mathcal{C}(X) \to \mathcal{C}(X)$ be a chain map. Let $T_k(X)$ denote the torsion subgroup of $H_k(X)$. Prove that $\varphi_k(T_k(X)) \subset T_k(X)$.

10.8 Let $X := \Gamma^1 = \operatorname{bd}[0,1]^2$. Compute the Lefschetz number of the map $f : X \to X$ given by

(a) $f(x_1, x_2) = (1 - x_1, x_2)$.
(b) $f(x_1, x_2) = (1 - x_1, 1 - x_2)$.

If $L(f) \neq 0$, explicitly find fixed points of f. If $L(f) = 0$, show that f has no fixed points. One should be aware of the fact that the reverse implication of Theorem 10.43 is not true in general.

10.9 * Let X be a cubical set, $F : X \rightrightarrows X$ a lower semicontinuous cubical acyclic valued map. Then F admits a chain selector φ, so $F_* = \varphi_*$, and the Lefschetz number $L(F)$ is defined for F by the same formula as it was done for a continuous map. A *fixed point* of F is a point x such that

$$x \in F(x).$$

Prove that, if $L(F) \neq 0$, then F has a fixed point.
Hint: Extract arguments from the proof of Theorem 10.43.

10.5 Degree Theory

We finished the previous Section with a rather remarkable result: If X is a cubical set and $E(X) \neq 0$, then *any* flow on X must have at least one equilibrium. Furthermore, this information is captured by a single number even though the homology groups $H_*(X)$ could be far from trivial. One might try to argue that this is not surprising since at the heart of the proof of Theorem 10.49 is a single number, the Lefschetz number. However, at this level it should appear even more impressive since given a continuous map $f : X \to X$, if $H_*(X)$ is high dimensional then $f_* : H_*(X) \to H_*(X)$ is represented by large matrices.

As important as the Lefschetz number is it does have a serious limitation. Consider the linear differential equation (10.4) of Example 10.38. The associated flow φ is defined on all of $\mathbf{R}^2$, which is not a cubical set. At the same time, we cannot try to mimic the approach of restricting our attention to a cubical set of the form $[-m, m]^2$, since for any $t > 0$, $\varphi_t([-m, m]^2) \not\subset [-m, m]^2$.

Obviously we would like to be able to circumvent this problem while still maintaining the simplicity of having a single number which can be used to guarantee the existence of equilibria. The proof of Theorem 10.49 is indirect in the sense that equilibria are obtained by analyzing the fixed points of the time τ maps of the flow for all positive τ. If we are working with a differential equation

$$\dot{x} = f(x) \quad x \in \mathbf{R}^d$$

then the direct question is:

> *does the equation $f(x) = 0$ have a solution in a prescribed set $U \subset \mathbf{R}^d$?*

Of course, there is an obvious relationship between fixed points of maps and zeros of a function. In particular, if $g : X \to X$ then

$$g(x) = x \quad \Leftrightarrow \quad f(x) := g(x) - x = 0.$$

With these observations in mind we turn to the topic of this section which is the degree of a map. In particular, it provides a single number that, if nonzero guarantees the existence of a solution to $f(x) = 0$ in a prescribed region U. However, a necessary first step is to understand the degree of maps on spheres.

10.5.1 Degree on Spheres

Consider a continuous map $f : S_0^d \to S_0^d$. Recall that the reduced homology groups of spheres (see Exercise 9.10) are

$$\tilde{H}_k(S_0^d) \cong \begin{cases} \mathbf{Z} & \text{if } k = d \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\tilde{H}_d(S_0^d) = \mathbf{Z}e$$

where e is a generator which is the equivalence class of a certain d-cycle z in $Z_d(S_0^d)$. Obviously there are only two choices of generators of $\mathbf{Z}e$: e or $-e$. Now the d-th homology map of f,

$$f_{*d} : \tilde{H}_d(S_0^d) \to \tilde{H}_d(S_0^d)$$

acting on the generator e must take the form

$$f_{*d}(e) = ne \tag{10.16}$$

where n is some integer. Moreover, that integer would be the same if we replace e by $-e$ because

$$f_{*d}(-e) = -f_{*d}(e) = -ne = n(-e).$$

Therefore the number n completely characterizes f_{*d} and is independent of the choice of generator of $\tilde{H}_d(S_0^d)$.

Definition 10.50 Given a continuous map $f : S_0^d \to S_0^d$, the *degree* of f denoted by $\deg(f)$ is the number n given by (10.16).

Proposition 10.51 *Degree has the following properties.*

(a) $\deg(id_{S_0^d}) = 1$*;*
(b) $\deg(c) = 0$*, where c is a constant map;*
(c) $\deg(f \circ g) = \deg(f)\deg(g)$*, where $f, g : S_0^d \to S_0^d$;*
(d) If $f \sim g$ then $\deg(f) = \deg(g)$*;*
(e) If f can be extended to a continuous map $\tilde{f} : \bar{B}_0^{d+1} \to S_0^d$ such that $\tilde{f}(x) = f(x)$ for $x \in S_0^d$ then $\deg(f) = 0$.

Proof. The conclusions are immediate from the material presented in Chapter 6:
(a) follows from Proposition 6.40;
(b) follows from the definition of the homology of a map;
(c) follows from Theorem 6.58;
(d) follows from Corollary 6.67.
(e) We have $f = \tilde{f} \circ j$ where $j : S_0^d \to \bar{B}_0^{d+1}$ is the inclusion map. Then $f_* = \tilde{f}_* \circ j_*$ by Theorem 6.58. But $\bar{B}_0^{d+1}$ is acyclic, so $j_{*d} = 0$ hence $f_{*d} = 0$. In the case $d = 0$, we use this argument for maps on reduced homology groups: $j_{*0} : \tilde{H}_0^0(S_0^0) \to \tilde{H}_0^0(\bar{B}_0^1) = 0$. □

While our derivation of the definition of degree was extremely efficient it provides no insight into the topological meaning of degree. The next example is meant to remedy this.

Example 10.52 Consider a parameterization of the unit circle $S_0^1 = \operatorname{bd}[-1,1]^2$ by $t \in [0,8]$ which represents counterclockwise winding of the interval $[0,8]$ around the circle, starting from $(-1,1)$ as the image of $t=0$ and ending at the same point at $t = 8$. This construction is analogous to what has been done for the boundary of the unit square in Section 5.4 and in Example 6.37. The explicit formula is

$$x(t) := \begin{cases} (-1,-1) + t(1,0) & \text{if } t \in [0,2], \\ (1,-1) + (t-2)(0,1) & \text{if } t \in [2,4], \\ (1,1) + (t-4)(-1,0)) & \text{if } t \in [4,6], \\ (-1,1) + (t-6)(0,-1) & \text{if } t \in [6,8]. \end{cases}$$

The above formula provides a continuous bijection of the interval $[0,8]$, with its endpoints identified together, onto S_0^1. Now let $f : S_0^1 \to S_0^1$ be given by

$$f(x(t)) := \begin{cases} x(2t) & \text{if } t \in [0,4], \\ x(2t-4) & \text{if } t \in [4,8]. \end{cases}$$

By the discussion in Section 5.4, f can be expressed in terms of parameter t as $t \mapsto 2t$ provided we make identification $2t \sim 2t-4$. Hence f is a doubling map on S_0^1. We interpret this geometrically as wrapping the circle twice in the counterclockwise direction. The calculation of M_f and its chain selector ϕ is left as Exercise 10.12. Note that rescaling is not necessary here. Once those steps are completed, it is easy to verify that $\phi_1(z) = 2z$ for the generating cycle

$$\begin{aligned} z = & \left(\widehat{[-1,0]} + \widehat{[0,1]}\right) \diamond \widehat{[-1]} + \widehat{[1]} \diamond \left(\widehat{[-1,0]} + \widehat{[0,1]}\right) \\ & - \left(\widehat{[-1,0]} + \widehat{[0,1]}\right) \diamond \widehat{[1]} - \widehat{[1]} \diamond \left(\widehat{[-1,0]} + \widehat{[0,1]}\right). \end{aligned}$$

It follows that $\deg(f) = 2$.

It is now easy to guess that if we define, in a similar manner, a map expressed by $t \mapsto nt$ where n is any integer then the induced degree should be n. For $|n| > 2$, however, one needs a rescaling in order to compute the homology map.

Observe that the degree is closely related to the Lefschetz number (see Definition 10.42):

Proposition 10.53 Let $f : S_0^d \to S_0^d$ be a continuous function and $d \geq 1$. Then

$$L(f) = 1 + (-1)^d \deg(f). \tag{10.17}$$

Proof. Since $H_k(S_0^d) = 0$ unless $k = 0$ or $k = d$, Definition 10.42 gives

$$L(f) = (-1)^0 \operatorname{tr} f_{*0} + (-1)^d \operatorname{tr} f_{*d} = \operatorname{tr} f_{*0} + (-1)^d \deg(f).$$

By Example 6.27 f induces the identity in $H_0(S_0^d)$, so the first term in the sum is 1. □

Example 10.54 Consider the *antipodal map* $-\mathrm{id} : S_0^d \to S_0^d$ which sends any x to its *antipode* $-x$. Clearly, the antipodal map has no fixed points on the sphere so by the Lefschetz Fixed Point Theorem 10.43, $L(-\mathrm{id}) = 0$. Thus, by Proposition 10.53

$$\deg(-\mathrm{id}_{S_0^d}) = (-1)^{d+1}.$$

The material presented above can serve to derive many interesting conclusions on fixed points of maps on odd and even dimensional spheres. By extending these results to euclidean spheres S_2^d with the tools presented in Chapter 11, one can also derive the famous *Poincaré-Brouwer Theorem* saying that there is no continuous non-vanishing tangent field on S_2^{2n}. Such fields exist on S_2^{2n-1}. We refer the reader to [15, 37].

Exercises

10.10 (a) Let $f, g : X \to S_0^d$ be two continuous maps. If $f(x) \neq -g(x)$ for all $x \in X$, show that $f \sim g$.

(b) Assume next that $X = S_0^d$. Deduce from (a) that if $f(x) \neq -x$ for all $x \in X$ then $f \sim 1_{S_0^d}$, and if $f(x) \neq x$ for all $x \in X$ then $f \sim -\mathrm{id}$ where $-\mathrm{id}$ is the *antipodal map* given by $-\mathrm{id}(x) = -x$.

10.11 * Here is an alternative way of proving that (c) implies (b) in Theorem 10.40. If $f : \bar{B}_0^{d+1} \to \bar{B}_0^{d+1}$ is fixed point free, the retraction $r : \bar{B}_0^{d+1} \to S_0^d$ can be defined as follows. For any $x \in \bar{B}_0^{d+1}$, since $f(x) \neq x$, there is a unique half-line emanating from $f(x)$ and passing through x. That open half-line intersects S_0^d et exactly one point $r(x)$. If $x \in S_0^d$ then obviously $r(x) = x$. Complete the proof by showing that r is continuous.

10.12 Complete the arguments in Example 10.52 by computing the multivalued representation of f and its chain selector.

10.13 Consider the symmetry $f_i : S_0^d \to S_0^d$ sending $x = (x_1, x_2 \dots x_{d+1})$ to the point y with coordinates $y_j = x_j$ if $j \neq i$ and $y_i = -x_i$. Prove that $\deg(f_i) = -1$,

(a) in the case $d = 1$;
(b) * in the general case.

10.14 Consider the coordinate exchange map $f_{i,i+1} : S_0^d \to S_0^d$ sending $x = (x_1, x_2 \dots x_{d+1})$ to the point y with coordinates $y_i = x_{i+1}$, $y_{i+1} = x_i$ and $y_j = x_j$ if $j \neq i, i+1$. Prove that $\deg(f_{i,i+1}) = -1$,

(a) in the case $d = 1$;
(b) * in the general case.

10.15 Let $f : S_0^d \to S_0^d$ be a given continuous map.

(a) If $\deg(f) \neq (-1)^{d+1}$, show that f must have a fixed point.
(b) If $\deg(f) \neq 1$, show that f must send a point x to its antipode $-x$.

10.5.2 Topological Degree

The reader may be somewhat bewildered at this point. In the introduction to this section we motivated degree by stating that it would be used to prove the existence of solutions to $f(x) = 0$. Obviously, if $f : S_0^d \to S_0^d$ then for any $x \in S_0^d$, $f(x) \neq 0$. We will now resolve this apparent non sequiter.

Consider the open subset U of R^d given by

$$U := B_0^d(0, m) = (-m, m)^d$$

where $m > 0$ is a given positive integer. Then U is a representable set whose closure

$$\operatorname{cl} U = \bar{B}^d(0, m) = [-m, m]^d$$

and whose boundary in $\mathbf{R}^d$ (see Definition 12.27)

$$\operatorname{bd} U = S_0^{d-1}(0, m) = \{x \in \operatorname{cl} U \mid x_i \in \{-m, m\} \text{ for some } i = 1, 2, \dots d\}$$

are cubical sets.

We are finally in a position to return to the original problem of finding solutions to

$$f(x) = 0 \tag{10.18}$$

for a continuous map $f : \operatorname{cl} U :\to R^d$ such that

$$f(x) \neq 0 \text{ for all } x \in \operatorname{bd} U. \tag{10.19}$$

Solutions x of (10.18) are called *roots* of f. The reason for imposing the condition (10.19) is that roots on the boundary of the domain are sensitive to small perturbations of f and cannot be detected by topological methods, as the following example illustrates.

Example 10.55 Consider $U = (-1, 1) \subset \mathbf{R}$ and a continuous function $f : [-1, 1] \to \mathbf{R}$. Put $a = f(-1)$ and $b = f(1)$. We examine the following cases.
Case 1. $a, b \neq 0$ and of opposite signs, say $a < 0$ and $b > 0$. By the intermediate value theorem, f must have a root in U. Moreover any sufficiently small perturbation f_ϵ of f has the same property. Indeed, it is enough to choose

$$\epsilon \leq \frac{1}{2} \min\{|a|, |b|\}.$$

If $|f_\epsilon(x) - f(x)| \leq \epsilon$, we see that $f_\epsilon(-1) < 0$ and $f_\epsilon(1) > 0$, so the Intermediate Value Theorem can be applied to f_ϵ. We will later see that this is a very simple case of a map with non zero degree.
Case 2. $a, b \neq 0$ have the same sign, say, $a, b > 0$. Then f may have no roots in U. For example if $f(x) := 1$, f has no roots. If $f(x) := x^2$, it has a root $x = 0$ but it would disappear if we consider a small perturbation $f_\epsilon = x^2 + \epsilon$. We will later see that this is a very simple case of a map of degree zero.
Case 3. $a = 0$ or $b = 0$, so assume $a = 0$ and $b > 0$. Consider $f(x) = x + 1$. Then f has a root $x = -1$. A small perturbation $f_\epsilon = x + 1 + \epsilon$ of f admits no roots. Another small perturbation $f_{-\epsilon} = x + 1 - \epsilon$ falls into Case 1, so not only it has a root but also it has the stability property described there.

We will reduce the problem of detecting roots of f to the study of maps in the unitary sphere S_0^{d-1} of the previous section. Define $\bar{f} : S_0^{d-1} \to S_0^{d-1}$ as follows.

$$\bar{f}(x) := \frac{f(mx)}{||f(mx)||_0}. \tag{10.20}$$

Note that $mx \in \operatorname{bd} U$ if and only is $x \in S_0^{d-1}$, so (10.19) implies that $\bar{f}$ is well defined.

Definition 10.56 Let $f : \operatorname{cl} U \to \mathbf{R}^d$ be a continuous map satisfying (10.19). Then the *topological degree* or, shortly, the *degree* of f on U is defined by the formula

$$\deg(f, U) := \deg(\bar{f}) \tag{10.21}$$

where $\deg(\bar{f})$ is given by Definition 10.50.

Here is the theorem relating the degree to detection of roots.

Theorem 10.57 *Let $f : \operatorname{cl} U \to \mathbf{R}^d$ be a continuous map satisfying (10.19). If $\deg(f, U) \neq 0$, then (10.18) has a solution in U.*

Proof. We argue by contradiction. Suppose that

$$f(x) \neq 0$$

for all $x \in \operatorname{cl} U = \bar{B}_0^d(0, m)$. Then

$$f(mx) \neq 0$$

for all x in the closed unitary ball $\bar{B}_0^d$, so the formula 10.20 can be extended to all $x \in \bar{B}_0^d$. By Proposition 10.51, $\deg(f,U) = \deg(\bar{f}) = 0$. □

We shall now discuss the homotopy invariance of degree analogous to the one established for the degree on spheres. Since we assume that f has no roots on $\operatorname{bd} U$ it is natural to expect that the same condition should be assumed for the map h_t where $h_t(x) = h(x,t)$ and $h : \operatorname{cl} U \times [0,1] \to R^d$ is a homotopy. To make this more transparent, we will rephrase (10.19) in terms of maps on pairs of spaces. We consider continuous maps

$$f : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\}).$$

That means that $f : \operatorname{cl} U \to \mathbf{R}^d$ has the property $f(\operatorname{bd} U) \subset \mathbf{R}^d \setminus \{0\}$, which is equivalent (10.19).

Definition 10.58 Two maps $f, g : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d - \{0\})$ are *homotopic on pairs* if there exists a continuous map

$$h : (\operatorname{cl} U \times [0,1], \operatorname{bd} U \times [0,1]) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$$

such that $h_0(x) = f(x)$ and $h_1(x) = g$ for all $x \in \operatorname{cl} U$.

Proposition 10.59 *Let* $f, g : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$ *be homotopic on pairs. Then* $\deg(f,U) = \deg(g,U)$.

Proof. If $h : (\operatorname{cl} U \times [0,1], \operatorname{bd} U \times [0,1]) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$ is a homotopy on pairs joining f to g, then the map $\bar{h} : S_0^d \to S_0^d$ defined by

$$\bar{h}(x,t) := \frac{h(mx,t)}{||h(mx,t)||_0}.$$

is a continuous homotopy on S_0^d. The conclusion now follows from Proposition 10.51[d]. □

The following proposition links homotopy to stability under perturbation.

Proposition 10.60 *Let* $f : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$ *be a continuous map. Then there exists an* $\epsilon > 0$ *such that if* $f_\epsilon : \operatorname{cl} U \to \mathbf{R}^d$ *has the property*

$$||f_\epsilon(x) - f(x)||_0 \leq \epsilon$$

then $f_\epsilon : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$ *and* $\deg(f_\epsilon) = \deg(f)$.

Proof. Since $\operatorname{bd} U$ is compact, and the continuous function $x \mapsto ||f(x)||_0$ has no zero on $\operatorname{bd} U$, it follows that

$$c := \inf\{||f(x)||_0 \mid x \in \operatorname{bd} U\} > 0.$$

It is enough to take any $\epsilon < c$ and the homotopy given by

$$h(x,t) := (1-t)f(x) + tf_\epsilon(x).$$

We get

$$||h(x,t) - f(x)||_0 = || - tf(x) + tf_\epsilon(x)||_0 \le t||f(x) - f_\epsilon(x)||_0 \le \epsilon.$$

If $h(x,t) = 0$ for some $(x_0, t_0) \in \operatorname{bd} U \times [0,1]$ then we get $||f(x_0)||_0 \le \epsilon$ which contradicts the choice of ϵ. Thus h is a homotopy on pairs and the conclusion follows from Proposition 10.59. □

We finish with a series of examples heading towards various applications of degree.

Example 10.61 Consider the map $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x_1, x_2) := (x_1^2 - x_2^2, 2x_1x_2). \tag{10.22}$$

We will show that $\deg(f, B_0^2) = 2$. We pass to polar coordinates

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$$

where $r \ge 0$ and $\theta \in [0, 2\pi]$ are referred to as *radius* and, respectively *argument* of $x = (x_1, x_2)$. By double angle formulas we get

$$f(r\cos\theta, r\sin\theta) = r^2(\cos 2\theta, \sin 2\theta)$$

The related map $\bar{f} : S_0^1 \to S_0^1$ is given by

$$\bar{f}(r\cos\theta, r\sin\theta) = \frac{(\cos 2\theta, \sin 2\theta)}{\max\{|\cos 2\theta|, |\sin 2\theta|\}}$$

so in terms of the argument θ it is a doubling map $\theta \mapsto 2\theta$ upon identification $\theta + 2\pi \sim \theta$. Note that the argument values

$$\theta = k\frac{\pi}{4}\ ,\ k = 0, 1, 2, \ldots 8$$

correspond to elementary vertices in S_0^1 and $\bar{f}$ maps them to vertices corresponding, respectively, to $2k\frac{\pi}{4}$. Intervals

$$\theta \in [k\frac{\pi}{4}, (k+1)\frac{\pi}{4}]$$

correspond to elementary intervals in S_0^1 and $\bar{f}$ maps them to pairs of two adjacent elementary intervals corresponding to

$$[2k\frac{\pi}{4}, (2k+1)\frac{\pi}{4}] \cup [(2k+1)\frac{\pi}{4}, (2k+2)\frac{\pi}{4}].$$

The map $\bar{f}$ is very similar to the map discussed in Example 10.52 except for two points: One, the loop there starts at the vertex $(-1,-1)$ and here at

$(1,0)$. Two, the map in Example 10.52 is linear on each elementary interval and here is not. But in the light of the above discussion, the multivalued representation and its chain selector can be calculated in the same way. After some calculation we get

$$\deg(f, B_0^2) = \deg(\bar{f}) = 2.$$

The formula 10.22 defining f looks complicated but it can be demystified by the use of complex numbers. Any $x = (x_1, x_2) \in \mathbf{R}$ is identified with the complex number $z = x_1 + \mathbf{i}x_2$ where $\mathbf{i} := \sqrt{-1}$. The complex multiplication gives

$$z^2 = x_1^2 - x_2^2 + \mathbf{i}2x_1x_2 = f(x_1, x_2),$$

so our result reflects the fact that 0 is the root of z^2 of multiplicity two. We may now be tempted to guess that

$$\deg(z^n, B_0^2) = n.$$

This can be proved in a similar way but one needs to consider a homotopy joining the map on a circle given by $\theta \mapsto n\theta$ with the map given by $t \mapsto nt$ discussed in Example 10.52 and consider an appropriate rescaling.

As the reader may already have observed it is easy to check that $f(0,0) = (0,0)$, and therefore using degree theory to prove the existence of a zero appears to be overkill. On the other hand, the same type of analysis used here leads to a proof of the Fundamental Theorem of Algebra (see Exercise 10.18).

Example 10.62 Let A be any $d \times d$ real matrix and let $f_A : \mathbf{R}^d \to \mathbf{R}^d$ be given by $f_A(x) := Ax$. Assume that A is nonsingular, i.e. $\det A \neq 0$. Then 0 is the only root of f_A in $\mathbf{R}^d$, so the degree of f_A is well defined on $U = B_0(0, m)$ for any value of m and

$$\deg(f_A, U) = \operatorname{sgn}(\det A) = (-1)^{\det A}. \tag{10.23}$$

Indeed, by linearity of f_A and homogeneity of the norm, the related map on the unit sphere is

$$\bar{f}(x) = \frac{f_A(mx)}{||f_A(mx)||_0} = \frac{Ax}{||Ax||_0}.$$

Any real matrix can be expressed as a product of real elementary matrices analogous to those studied in Chapter 3 in the integer case:

- $E_{i,i+1}$ obtained by exchange of rows i and $i+1$ in the identity matrix,
- $E_i(c)$, obtained by multiplication of row i by $c \in \mathbf{R}$, and
- $E_{i,j,q}$ obtained by adding row j times q to row i, $i \neq j$, $q \in \mathbf{R}$.

By the product formula in Proposition 10.51 and the analogous product formula for determinants, the problem is reduced to finding the degree of an elementary matrix.

From Exercise 10.14 we get

$$\deg(f_{E_{i,i+1}}, U) = -1.$$

From Exercise 10.13 we get

$$\deg(f_{E_i(c)}, U) = -\operatorname{sgn} c.$$

Finally, the map $h : (\operatorname{cl} U \times [0,1], \operatorname{bd} U \times [0,1]) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\})$ given by

$$h(x,t) := E_{i,j,(1-t)q}x$$

is a homotopy joining $f_{E_{i,j,q}}$ to the identity map, hence

$$\deg(f_{E_{i,j,q}}, U) = 1.$$

Now the formula follows from Proposition 10.51(c) and the analogous product formula for determinants.

Example 10.63 Consider the differential equation

$$\dot{x} = f(x)\ ,\ x \in R^d, \tag{10.24}$$

where $f : \mathbf{R}^d \to \mathbf{R}^d$ is a continuous map such that the associated flow is well defined on $\mathbf{R}^d \times \mathbf{R}$.

Let A be a nonsingular real $d \times d$ matrix and suppose that f satisfies the condition

$$||f(x) - Ax||_0 \leq c \text{ for all } x \in \mathbf{R}^d.$$

for some $c > 0$. We show that Equation 10.24 has an *equilibrium solution* given by $x(t) := \bar{x}$ for all t. This is equivalent to showing that there exist $\bar{x} \in \mathbf{R}^d$ such that $f(\bar{x}) = 0$. By the assumption on A, the number

$$M = \min\{||Ax||_0 \mid x \in S_0^{d-1}\}$$

is strictly positive. Choose an integer $m > M^{-1}c$. It is easy to show that the equation $f(x) = 0$ cannot have solutions on $\operatorname{bd} U$ for $U = (-m,m)^d$ so $\deg(f,U)$ is well defined. Moreover, by the same arguments as in the proof of Proposition 10.60, f is homotopic to to f_A given by the matrix A, so by Example 10.62,

$$\deg(f,U) = \deg(f_A) = (-1)^{\det A}$$

and the conclusion follows. Note that we also found a bound for the norm of a possible stationary point, $||\bar{x}||_0 < M^{-1}c$.

Consider a continuous map

$$f : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbf{R}^d, \mathbf{R}^d \setminus \{0\}),$$

where $U = B_0(0, m)$. As we mentioned in the introduction to this section, the equation $f(x) = 0$ is equivalent to the fixed point equation $g(x) = x$ where $g(x) = x - f(x)$. The condition $f(x) \neq 0$ on $\operatorname{bd} U$ is equivalent to saying that g does not have fixed points on $\operatorname{bd} U$. The presence of fixed points of g in U can be detected by the inequality

$$\deg(\operatorname{id}_{\operatorname{cl} U} - g, U) \neq 0.$$

Many interesting applications of the degree theory to nonlinear ordinary and partial differential equations require generalizing the degree to infinite dimensional spaces of functions. The reader interested in pursuing this topic is referred to [15, 31].

Exercises

10.16 Let $U := (-1, 1)$, and let $f : [-1, 1] \to \mathbf{R}$ be any continuous map such that $f(t) \neq 0$ for $t = -1, 1$. Show that $\deg(f, U)$ is either 0, 1 or -1. Interpret your answer graphically.

10.17 Let $U = (-m, m)^2$. By taking for granted that $\deg(z^n, U) = n$ where z^n is the complex polynomial discussed in Example 10.61, construct a map $f : [-m, m]^2 \to \mathbf{R}^2$ with precisely n distinct roots in U and $\deg(f, U) = n$.

10.18 Consider $U = (-m, m)^2$ as in the previous exercise and the complex polynomial

$$f(z) := z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0,$$

where α_i are real or complex coefficients. Again, by taking for granted that $\deg(z^n, U) = n$, show that $\deg(f, U) = n$ for sufficiently large m. Conclude that f must have a root in the plane. This conclusion is known as *Fundamental Theorem of Algebra*.

10.6 Complicated Dynamics

As was mentioned in the introduction one of the goals of this Chapter is to show that the algebraic topological tools being developed can be combined with standard numerical techniques to provide rigorous proofs about the dynamics of nonlinear systems. In the previous section it was shown that homology can be used to prove the existence of invariant sets. However, the mathematical justification relied on the Ważewski principle which is not the ideal tool to combine with rigorous numerical computations. This can been seen from Example 10.39 where the existence of an invariant set for a particular system does not imply the existence of an invariant set for a perturbed system. Of course, because of floating point errors and discretization errors we can never expect numerical computations to exactly reproduce the nonlinear system being simulated. Therefore, the tools needed for computer assisted proofs must have the feature that they guarantee the existence of invariant sets not only for a particular flow but also for the nearby flows. Obviously, this feature by itself will not prove anything about the particular system from the study of the nearby system obtained by means of numerical computations. Nevertheless, without such a feature there is no hope for any computer assisted proof. This means that we need to replace the Ważewski Principle by something which persists under perturbations.

Another limitation of the Ważewski principle as described in the previous section is that it is not applicable to maps. Recall that the proof of Theorem 10.34 makes essential use of the flow to construct the deformation retraction. This theory is applicable both to flows and to maps, and furthermore, it has the important property that it can be used for rigorous computation. As such it provides a powerful method for computer assisted proofs in dynamics. While a full development of the Conley index is beyond the scope of this book, our goal is to present some of the ideas that are essential to its effective computation. We will use the Ważewski principle to motivate the small part of the theory that we will present here. Since most numerical schemes result in maps (time discretization is a typical method for studying ordinary differential equations) we will present the ideas primarily in the context of continuous functions rather than flows. However we want to emphasize that there is a rigorous method which allows to go back from the computations for maps to the results for flows.

10.6.1 Index Pairs and Index Map

As was mentioned in the introduction one of the goals of this section is to demonstrate that the computer can be used to rigorously study the dynamics of nonlinear systems. This leads us to adopt the following philosophy. We are interested in the dynamics of a continuous map $f : \mathbf{R}^d \to \mathbf{R}^d$, but because we are using the computer to perform calculations we only use the information available in the form of combinatorial cubical multivalued maps

$$\mathcal{F} : \mathcal{K}_{\max} \rightrightarrows \mathcal{K}_{\max},$$

where $\mathcal{K}_{\max} := \mathcal{K}_{\max}(\mathbf{R}^d)$ is the set of all maximal (i.e. d-dimensional) elementary cubes in $\mathbf{R}^d$. Such maps were introduced in Definition 5.4 in the context of approximating continuous functions. In Chapter 6 the related lower semicontinuous maps $F = \lfloor \mathcal{F} \rfloor$ (see (6.6)) were used for defining homomorphisms induced by maps in homology.

At the risk of being redundant, observe that there are three objects that we will be working with. A continuous function f which generates the dynamical system that we are interested in studying. A combinatorial map $\mathcal{F}$ which is generated by and manipulated by means of a computer, and a multivalued topological map F which allows us to pass from the cominatorics of the computer to statements about the dynamics of f via homology. How this last step is performed is the subject of this section.

Clearly, to make use of such a strategy there needs to be an appropriate relationship between f and $\mathcal{F}$. To explain it, we recall the following notation. Given a finite subset $\mathcal{X} \in \mathcal{K}_{\max}(\mathbf{R}^d)$

$$|\mathcal{X}| := \bigcup \mathcal{X}.$$

Obviously $X = |\mathcal{X}|$ is a cubical set in $\mathbf{R}^d$ (though of course not all cubical sets of $\mathbf{R}^d$ are of this form, they may be composed of lower dimensional elementary cubes). The set $\mathcal{X}$ can be recovered from X by the formula $\mathcal{X} = \mathcal{K}_{\max}(X)$.

Given any closed bounded subset X of $\mathbf{R}^d$, we will use the notation

$$\mathrm{o}(X) := \{Q \in \mathcal{K}_{\max} \mid Q \cap X \neq \emptyset\}.$$

The set $\mathrm{o}(X) \subset \mathcal{K}_{\max}$ will be called a *collar* of X. This notation and terminology is next extended to collars of points $x \in \mathbf{R}^d$ by putting

$$\mathrm{o}(x) := \mathrm{o}(\{x\})$$

and to collars of finite collections of elementary cubes $\mathcal{X}$ in $\mathbf{R}^d$ by putting

$$\mathrm{o}(\mathcal{X}) := \mathrm{o}(|\mathcal{X}|).$$

It is left as an exercise to verify that, given a cubical set X, $|\mathrm{o}(X)|$ is equal to $\bar{B}_1(X)$, the closed cubical ball of radius 1 about X, defined in Exercise 6.3.

Definition 10.64 A combinatorial multivalued map $\mathcal{F} : \mathcal{K}_{\max} \rightrightarrows \mathcal{K}_{\max}$ is a *combinatorial enclosure* of $f : \mathbf{R}^d \to \mathbf{R}^d$ if for every $Q \in \mathcal{K}_{\max}$

$$\mathrm{o}(f(Q)) \subset \mathcal{F}(Q).$$

We shall need the following characterization of topological interiors of sets in terms of collars of points:

Proposition 10.65 *Assume $\mathcal{X} \subset \mathcal{K}_{max}(\mathbf{R}^d)$ is a finite subset. Then*

$$\operatorname{int}|\mathcal{X}| = \left\{x \in \mathbf{R}^d \mid o(x) \subset \mathcal{X}\right\}.$$

Proof. To prove that the left hand side is contained in the right hand side take $x \in \operatorname{int}|\mathcal{X}|$ and assume that there is a $Q \in \mathrm{o}(x) \setminus \mathcal{X}$. By Proposition 2.15(iv) we can choose a sequence $\{x_n\} \subset \overset{\circ}{Q}$ such that $x_n \to x$. One can show that by Proposition 2.15(vi) $\mathrm{o}(x_n) = \{Q\}$. Therefore $x_n \notin |\mathcal{X}|$ and consequently $x \notin \operatorname{int}|\mathcal{X}|$.

To show the opposite inclusion assume $x \in \mathbf{R}^d$ is such that $\mathrm{o}(x) \subset \mathcal{X}$. Then by Proposition 6.10(i)

$$x \in \mathrm{oh}\,(x) \subset |\mathrm{o}(x)| \subset |\mathcal{X}|$$

and since $\mathrm{oh}\,(x)$ is open by by Proposition 6.10(ii), it follows that $x \in \mathrm{int}\,|\mathcal{X}|$
□

As an immediate corollary of this proposition we have the following alternative condition for a combinatorial enclosure.

Proposition 10.66 *A combinatorial multivalued map $\mathcal{F} : \mathcal{K}_{max} \rightrightarrows \mathcal{K}_{max}$ is a combinatorial enclosure of $f : \mathbf{R}^d \to \mathbf{R}^d$ if for every $x \in \mathbf{R}^d$ and $Q \in o(x)$*

$$f(Q) \subset \mathrm{int}\,(|\mathcal{F}(Q)|).$$

Proposition 10.66 is helpful in understanding the topology behind Definition 10.64 but the algorithm constructing a combinatorial enclosure of a given map is directly obtained from Definition 10.64. As in Chapter 7, we restrict the statement of the algorithm to rational maps.

Algorithm 10.67 Combinatorial Enclosure

```
function combinatorialEnclosure(set X, rationalMap f)
for each Q in X do
    A := evaluate(f, Q);
    U := collar(A);
    F{Q} := U;
endfor;
return F;
```

Presentation of the algorithm `collar` computing $\mathrm{o}(X)$ where X is a cubical set is left as an exercise.

Note that the data structure `cubicalSet` makes no distinction between a cubical set X and the finite set $\mathcal{X} = \mathcal{K}_{\max}(X)$ so the algorithm `collar` will equally apply to sets $\mathcal{X} \in \mathcal{K}_{\max}$.

Having established the relationship between f and $\mathcal{F}$ we now turn to the question of how homology can be used to understand the dynamics of f. As was discussed in the introduction, our goal is to introduce a topological object that is analogous to the Ważewski principle and robust with respect to perturbations. We have dealt with the question of sensitivity to perturbations when we introduced topological degree (see Example 10.55). In that setting we imposed the condition that the function could not be zero on the boundary of the set of interest. The following definition captures the same idea, but in the more general context of invariant sets.

Definition 10.68 A compact set $N \subset X$ is an *isolating neighborhood* for a continuous map f if

$$\mathrm{Inv}\,(N, f) \subset \mathrm{int}\,(N).$$

An invariant set S is *isolated* under f if there exists an isolating neighborhood N such that

$$S = \text{Inv}\,(N, f).$$

The same definition applies to flows by replacing f by φ.

The following proposition, which is essentially a restatement of the definition, provides another characterization of the isolating neighborhood.

Proposition 10.69 *A compact set N is an isolating neighborhood for f if and only if for every x in the boundary N there exists no full solution $\sigma_x : \mathbf{Z} \to N$.*

Returning to the Ważewski principle for motivation, recall that given a flow we need a pair of spaces (W, W^-) where W is a Ważewski set and W^- is the set of points which leave W immediately. The analogous conditions for the setting of maps are as follows.

Definition 10.70 Given a continuous map $f : X \to X$, an *index pair* consists of a pair of compact sets $P = (P_1, P_0)$ where $P_0 \subset P_1$ satisfying the following three properties:

1. *(isolation)* The set $\text{cl}\,(P_1 \setminus P_0)$ is an isolating neighborhood for f;
2. *(positive invariance)* If $x \in P_0$ and $f(x) \in P_1$ then $f(x) \in P_0$;
3. *(exit set)* If $x \in P_1$ and $f(x) \notin P_1$, then $x \in P_0$.

Because of the last property P_0 is referred to as the *exit set* for the pair.

The following example may appear to be too trivial to be worth presenting, however it will re-appear in a crucial manner at the end of this subsection.

Example 10.71 Observe that $\emptyset$ is a cubical set. Furthermore, given any continuous function f,

$$\emptyset = \text{Inv}\,(\emptyset, f) \subset \text{int}\,(\emptyset).$$

Therefore, $\emptyset$ is an isolated invariant set.

As was suggested earlier index pairs are analogues of Ważewski sets and P_0 plays a role similar to that of W^-. Therefore, one might hope that if $H_*(P_1, P_0) \not\cong 0$, then $\text{Inv}\,(N, f) \neq \emptyset$. Unfortunately, as the following example indicates this is wrong.

Example 10.72 Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x + 1$. Let $N = \{0\} \cup [1, 2]$. Clearly, $\text{Inv}\,(N, f) = \emptyset$. Let $(P_1, P_0) = (N, [1, 2])$. We leave it to the reader to check that this is an index pair. Observe that

$$H_k(P_1, P_0) = H_*(\{0\} \cup [1, 2], [1, 2]) \cong \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This example suggests that we need a more refined approach if we wish to use algebraic topology to capture the dynamics. As will be shown later what is missing is information that is available in the map f. However, since our philosophy is that it is the combinatorial map $\mathcal{F}$ and not f that is known we need to think about dynamics on the combinatorial level.

The basic definitions related to the dynamics of a map can be carried over to combinatorial maps, but we must remember that the variables are now elementary cubes and not points in $\mathbf{R}^d$.

Definition 10.73 A full solution through $Q \in \mathcal{K}_{\max}$ under $\mathcal{F}$ is a function $\mathit{\Gamma}_Q : \mathbf{Z} \to \mathcal{K}_{\max}$ satisfying the following two properties:

1. $\mathit{\Gamma}_Q(0) = Q$,
2. $\mathit{\Gamma}_Q(n+1) \in \mathcal{F}(\mathit{\Gamma}_Q(n))$ for all $n \in \mathbf{Z}$.

Definition 10.74 Let $\mathcal{N}$ be a finite subset of $\mathcal{K}_{\max}$. The *maximal invariant set* in $\mathcal{N}$ under $\mathcal{F}$ is

$$\operatorname{Inv}(\mathcal{N}, \mathcal{F}) := \{Q \in \mathcal{N} \mid \text{there exists a full solution } \mathit{\Gamma}_Q : \mathbf{Z} \to \mathcal{N}\}.$$

Although maximal invariant sets under continuous maps are usually not explicitly known and they are the subject of our investigation, their combinatorial counterparts are the ones which can be easily computed. We give below an alternative characterization of $\operatorname{Inv}(\mathcal{N}, \mathcal{F})$ which leads to an algorithm computing it. First, given a combinatorial map $\mathcal{F} : \mathcal{K}_{\max} \rightrightarrows \mathcal{K}_{\max}$, we would like to define its inverse $\mathcal{F}^{-1} : \mathcal{K}_{\max} \rightrightarrows \mathcal{K}_{\max}$ by the formula:

$$\mathcal{F}^{-1}(R) := \{Q \in \mathcal{K}_{\max} \mid R \in \mathcal{F}(Q)\}.$$

The two problems with this formula are that, one, the values of $\mathcal{F}^{-1}$ might be empty, two, they might contain infinitely many cubes which we don't want to. The first problem will not bother us but, because of the second, we should restrict the domain of $\mathcal{F}$ to a finite subset $\mathcal{X} \subset \mathcal{K}_{\max}$. Thus we define $\mathcal{F}_{\mathcal{X}} : \mathcal{X} \to \mathcal{K}_{\max}$ be the restriction of $\mathcal{F}$ to $\mathcal{X}$ given by $\mathcal{F}_{\mathcal{X}}(Q) := Q$ for all $Q \in \mathcal{X}$. The inverse of $\mathcal{F}_{\mathcal{X}}$ given by

$$\mathcal{F}_{\mathcal{X}}^{-1}(R) := \{Q \in \mathcal{X} \mid R \in \mathcal{F}(Q)\}.$$

for all $R \in \mathcal{K}_{\max}$ has finite values.

Theorem 10.75 (Szymczak [43]) *Let $\mathcal{F} : \mathcal{K}_{max} \rightrightarrows \mathcal{K}_{max}$ be a combinatorial map and let $\mathcal{N}$ be a finite subset of $\mathcal{K}_{max}$. Construct the sequence of subsets $\{\mathcal{S}_j\}_{j=0,1,2,\ldots}$ of $\mathcal{K}_{max}$ by induction as follows.*

$$\mathcal{S}_0 := \mathcal{N};$$
$$\mathcal{S}_{j+1} := \mathcal{F}(\mathcal{S}_j) \cap \mathcal{S}_j \cap \mathcal{F}_{\mathcal{N}}^{-1}(\mathcal{S}_j).$$

Then the sequence eventually becomes constant, i.e. there exists an index j_0 such that $\mathcal{S}_j = \mathcal{S}_{j_0}$ for all $j \geq j_0$. Moreover, for such j_0, we have

$$\operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \mathcal{S}_{j_0}$$

Proof. It is clear that the sequence of sets $\{\mathcal{S}_j\}$ is decreasing, i.e. $\mathcal{S}_{j+1} \subset \mathcal{S}_j$ for all $j = 0, 1, 2, \ldots$. Since $\mathcal{S}_0$ is finite, there must be an integer j_0 such that $\mathcal{S}_j = \mathcal{S}_{j_0}$ for all $j \geq j_0$, so the first statement is proved. To prove that $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \mathcal{S}_{j_0}$, first notice that we have

$$\mathcal{S}_{j_0} = \mathcal{S}_{j_0+1} = \mathcal{F}(\mathcal{S}_{j_0}) \cap \mathcal{S}_{j_0} \cap \mathcal{F}_{\mathcal{N}}^{-1}(\mathcal{S}_{j_0}).$$

This implies that for every element $Q \in \mathcal{S}_{j_0}$ there are elements $K \in \mathcal{S}_{j_0}$ and $L \in \mathcal{S}_{j_0}$ such that $Q \in \mathcal{F}(K)$ and $L \in \mathcal{F}(Q)$. The same arguments can be again applied to K and L. Thus a solution $\Gamma_Q : \mathbf{Z} \to \mathcal{K}_{\max}$ through Q under $\mathcal{F}$ can be defined by putting

$$\Gamma_Q(0) := Q \ , \ \Gamma_Q(-1) := K \ , \ \Gamma_Q(1) := L,$$

and applying induction twice, for $n = 1, 2, 3, \ldots$ and for $n = -1, -2, -3, \ldots$. Since $\mathcal{S}_{j_0} \subset \mathcal{N}$, the solution obviously has values in $\mathcal{N}$. Since Q is arbitrary in $\mathcal{S}_{j_0}$, this proves that $\mathcal{S}_{j_0} = \operatorname{Inv}(\mathcal{S}_{j_0}, \mathcal{F}) \subset \operatorname{Inv}(\mathcal{N}, \mathcal{F})$.

To prove the reverse inclusion, notice that

$$\mathcal{S}_{j_0} = \bigcap_{k=0}^{j_0} \mathcal{S}_k = \bigcap_{k=0}^{\infty} \mathcal{S}_k,$$

so it is enough to prove that $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{S}_k$ for all $k = 0, 1, 2, \ldots$. We will prove this by induction on k. By definition, $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{N} = \mathcal{S}_0$ so the hypothesis is true for $k = 0$. Let now $k \geq 0$ and suppose that $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{S}_k$. Then any $Q \in \operatorname{Inv}(\mathcal{N}, \mathcal{F})$ admits a solution $\Gamma_Q : \mathbf{Z} \to \operatorname{Inv}(\mathcal{N}, \mathcal{F})$ through Q under $\mathcal{F}$. Put $Q_n := \Gamma_Q(n)$ for $n \in \mathbf{Z}$. By the induction hypothesis, $Q_n \in \mathcal{S}_k$ for all n. We have $Q = Q_0 \in \mathcal{F}(Q_{-1})$ and $Q_1 \in \mathcal{F}(Q)$. The last identity is equivalent to $Q \in \mathcal{F}_{\mathcal{N}}^{-1}(Q_1)$. We get

$$Q \in \mathcal{F}(Q_{-1}) \cap \mathcal{F}_{\mathcal{N}}^{-1}(Q_1) \subset \mathcal{S}_{k+1}.$$

Thus $Q \in \mathcal{S}_{k+1}$ and we have proved that $\operatorname{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{S}_{k+1}$. $\square$

The above theorem guarantees that the following algorithm always stops and returns the set $\operatorname{Inv}(\mathcal{N}, \mathcal{F})$.

Algorithm 10.76 Combinatorial Invariant Set

```
function invariantPart(set N, combinatorialMap F)
F := restrictedMap(F, N);
Finv := evaluateInverse(F);
S := N;
repeat
    S' := S;
    S := intersection(S, evaluate(F, S));
    S := intersection(S, evaluate(Finv, S));
until (S = S');
return S;
```

The construction of algorithms `restrictedMap` and `evaluateInverse` evaluating, respectively, the restricted map $\mathcal{F}_N$ and the inverse map $\mathcal{F}^{-1}$ are left as exercises.

Definition 10.77 Let $\mathcal{N}$ be a finite subset of $\mathcal{K}_{\max}$. The set $\mathcal{N}$ is an *isolating neighborhood* for $\mathcal{F}$ if

$$\mathrm{o}(\mathcal{S}) \cup \mathcal{F}(\mathcal{S}) \subset \mathcal{N}. \tag{10.25}$$

Candidates for isolating neighborhoods have to be determined empirically. However, it is possible to check by a finite computation whether or not a given set $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$.

Algorithm 10.78 Isolation Property

```
function isolation(set N, combinatorialMap F) : boolean
S := invariantPart(N, F);
U := collar(S);
FS := evaluate(F, S);
if subset(union(U, FS), N) then
    return true;
else return false;
```

The following theorem indicates that using $\mathcal{F}$ we can construct index pairs for f.

Theorem 10.79 *Suppose that $\mathcal{F} : \mathcal{K}_{max} \rightrightarrows \mathcal{K}_{max}$ is a combinatorial enclosure of $f : \mathbf{R}^d \to \mathbf{R}^d$. Let $\mathcal{N} \in \mathcal{K}_{max}$ be an* isolating neighborhood *for $\mathcal{F}$ and let $\mathcal{S} := \operatorname{Inv}(\mathcal{N}, \mathcal{F})$. Define*

$$\mathcal{P}_1 := \mathcal{F}(\mathcal{S}) \text{ and } \mathcal{P}_0 := \mathcal{F}(\mathcal{S}) \setminus \mathcal{S}.$$

Then

(a) $|\mathcal{S}|$ is an isolating neighborhood for f.
(b) $(P_1, P_0) := (|\mathcal{P}_1|, |\mathcal{P}_0|)$ is an index pair for f.

Proof. We prove (a) by contradiction. Suppose that $|\mathcal{S}|$ is not an isolating neighborhood for f. This means that

$$\operatorname{Inv}(|\mathcal{S}|, f) \cap \operatorname{bd}|\mathcal{S}| \neq \emptyset,$$

so there exists a full solution $\gamma_x : \mathbf{Z} \to |\mathcal{S}|$ through a boundary point $x \in \operatorname{bd}|\mathcal{S}|$ under f. Then $x \notin \operatorname{int}|\mathcal{S}|$. Thus by Proposition 10.65 we can choose $Q \in \mathrm{o}(x) \setminus \mathcal{S}$. We define $\mathit{\Gamma}_Q : \mathbf{Z} \to \mathcal{K}_{\max}$ as follows. We put $\mathit{\Gamma}_Q(0) := Q$ and for $n \neq 0$, we let $\mathit{\Gamma}_Q(n)$ be any element of $\mathrm{o}(\gamma_x(n))$. Then by Definition 10.64

$$\mathit{\Gamma}_Q(n+1) \in \mathrm{o}(\gamma_x(n+1)) = \mathrm{o}(f(\gamma_x(n))) \subset \mathcal{F}(\mathit{\Gamma}_Q(n)),$$

which shows that $\mathit{\Gamma}_Q$ is a full solution through Q under $\mathcal{F}$. Moreover, since $\gamma_x(n) \in |\mathcal{S}| \cap \mathit{\Gamma}_Q(n)$, it follows from (10.25) that $\mathit{\Gamma}_Q(n) \in \mathrm{o}(\mathcal{S}) \subset \mathcal{N}$. Therefore $Q = \mathit{\Gamma}_Q(0) \in \operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \mathcal{S}$, which contradicts the choice of Q.

To prove (b), we must verify the three properties of index pairs in Definition 10.70. Before we start, let us make an observation which will serve us on several occasions. Since $\mathcal{S}$ is invariant under $\mathcal{F}$, it is contained in $\mathcal{F}(\mathcal{S})$, so we get

$$\mathcal{P}_1 \setminus \mathcal{P}_0 = \mathcal{F}(\mathcal{S}) \setminus (\mathcal{F}(\mathcal{S}) \setminus \mathcal{S}) = \mathcal{F}(\mathcal{S}) \cap \mathcal{S} = \mathcal{S}. \tag{10.26}$$

Put $\mathcal{R} := \mathcal{P}_1 \setminus \mathcal{P}_0$. We have

$$P_1 \setminus P_0 = |\mathcal{P}_1| \setminus |\mathcal{P}_0| = \bigcup_{Q \in \mathcal{P}_1} (Q \setminus |\mathcal{P}_0|) = \bigcup_{Q \in \mathcal{R}} (Q \setminus |\mathcal{P}_0|)$$

Therefore

$$\mathrm{cl}\,(P_1 \setminus P_0) = \bigcup_{Q \in \mathcal{R}} \mathrm{cl}\,(Q \setminus |\mathcal{P}_0|) = \bigcup_{Q \in \mathcal{R}} Q = |\mathcal{P}_1 \setminus \mathcal{P}_0|$$

and by (10.26)

$$\mathrm{cl}\,(P_1 \setminus P_0) = |\mathcal{S}|, \tag{10.27}$$

which shows that the isolation property (1) is already proved in (a). For proving the exit set property (2), note again that

$$\mathcal{P}_1 \setminus \mathcal{P}_0 = \mathcal{S} \subset \mathcal{F}(\mathcal{S})$$

which implies that

$$\mathcal{F}(\mathcal{P}_1 \setminus \mathcal{P}_0) \subset \mathcal{P}_1. \tag{10.28}$$

We will show that

$$f(P_1 \setminus P_0) \subset P_1. \tag{10.29}$$

Indeed, if $x \in P_1 \setminus P_0$ and $f(x) \notin P_1$, then there exists a $Q \in \mathrm{o}(f(x)) \setminus \mathcal{P}_1$. But by (10.27) $x \in |\mathcal{S}|$, i.e. there exists an $R \in \mathrm{o}(x) \cap \mathcal{S}$ and by Definition 10.64

$$Q \in \mathrm{o}(f(x)) \subset \mathcal{F}(R) \subset \mathcal{F}(\mathcal{S}) = \mathcal{P}_1,$$

which contradicts the choice of Q. Thus (10.29) is proved. It is easily verified that this condition is equivalent to the exit set property (2).

To prove the positive invariance property (3), we first prove its combinatorial counterpart

$$\mathcal{F}(\mathcal{P}_0) \cap \mathcal{P}_1 \subset \mathcal{P}_0. \tag{10.30}$$

We do this by contradiction. Suppose that there is a cube $Q \in \mathcal{P}_0$ such that

$$\mathcal{F}(Q) \cap (\mathcal{P}_1 \setminus \mathcal{P}_0) = \mathcal{F}(Q) \cap \mathcal{S} \neq \emptyset.$$

Choose $R \in \mathcal{F}(Q) \cap \mathcal{S}$. Since $\mathcal{S} = \mathrm{Inv}\,(\mathcal{N}, \mathcal{F})$, there is a full solution Γ_R through R with values in $\mathcal{N}$. Next, $Q \in \mathcal{P}_0 \subset \mathcal{F}(\mathcal{S})$ so there is a cube $R' \in \mathcal{S}$ with $Q \in \mathcal{F}(R')$. This cube R' has a full solution $\Gamma_{R'}$ under $\mathcal{F}$ with values in $\mathcal{N}$. We also have by (10.25) that $Q \in \mathcal{F}(R') \subset \mathcal{F}(\mathcal{S}) \subset \mathcal{N}$. This lets us define a full solution Γ_Q through Q under $\mathcal{F}$ with values in $\mathcal{N}$ by combining the negative-time part of Γ'_R with the positive-time part of Γ_R:

$$\Gamma_Q(n) = \begin{cases} \Gamma_R(n-1) & \text{if } n \geq 1 \\ Q & \text{if } n = 0 \\ \Gamma_{R'}(n+1) & \text{if } n \leq -1. \end{cases}$$

Since Γ_Q is a full solution under $\mathcal{F}$ contained in $\mathcal{N}$, we get $Q \in \operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \mathcal{S}$ in contrary to the hypothesis that $Q \in \mathcal{P}_0 = \mathcal{F}(\mathcal{S}) \setminus \mathcal{S}$.

Finally we are ready to prove the positivie invariance property (3). Let $x \in P_0$ and $f(x) \in P_1$. Then $x \in Q$ for some $Q \in \mathcal{P}_0$ and $f(x) \in R$ for some $R \in \mathcal{P}_1$. Thus, by Definition 10.64

$$R \in \mathrm{o}(f(x)) \subset \mathcal{F}(Q) \subset \mathcal{F}(\mathcal{P}_0)$$

and by (10.30) $R \in \mathcal{P}_0$, which proves that $f(x) \in P_0$. $\square$

Example 10.80 Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 2x$. Let $\mathcal{F} : \mathcal{K}_{\max}(\mathbf{R}) \rightrightarrows \mathcal{K}_{\max}(\mathbf{R})$ be given by

$$\mathcal{F}([i, i+1]) = \{[2i-1, 2i], [2i, 2i+1], [2i+1, 2i+2], [2i+2, 2i+3]\}.$$

Let $\mathcal{N} = \{[i, i+1] \mid i = -10, \ldots, 9\}$. It is left to the reader to check that

$$\mathcal{S} = \operatorname{Inv}(\mathcal{N}, \mathcal{F}) = \{[-2,-1], [-1,0], [0,1], [1,2]\}.$$

By Theorem 10.79 $(P_1, P_2) = ([-5,5], [-5,-2] \cup [2,5])$ is an index pair for f.

On a combinatorial level it is only the pair of finite sets

$$(\mathcal{P}_1, \mathcal{P}_0) = (\mathcal{F}(\mathcal{S}), \mathcal{F}(\mathcal{S}) \setminus \mathcal{S})$$

which needs to be constructed.

The following algorithm is obtained by combining Algorithms 10.67, 10.78, 10.76, and Theorem 10.79.

Algorithm 10.81 Combinatorial Index Pair

```
function indexPair(set N, rationalMap f)
F := combinatorialEnclosure(N,f);
if isolation(N, F) then
    S := invariantPart(N, F);
    FS := evaluate(F, S);
    Pone := FS;
    Pzero := setminus(FS, S);
    P := (Pone,Pzero);
    return P;
else
    return failure;
endif;
```

Unfortunately, unless $P_0 = \emptyset$, f may fail to be a self-map of (P_1, P_0) i.e. it may be the case that $f(P_1) \not\subset P_1$ or $f(P_0) \not\subset P_0$. Sticking to our computational philosophy all we know is how $\mathcal{F}$ acts on $(\mathcal{P}_1, \mathcal{P}_0)$ but $\mathcal{F}$ may also fail to be a self map.

Thus, as a first step we can try to determine an appropriate range for $\mathcal{F}$. Obviously, the minimal requirement is a pair of sets $(\mathcal{X}, \mathcal{A})$ in $\mathcal{K}_{\max}$ such that

$$\mathcal{F}(\mathcal{P}_1) \subset \mathcal{X} \text{ and } \mathcal{F}(\mathcal{P}_0) \subset \mathcal{A}.$$

Since $\mathcal{F}$ is an enclosure of f, this will imply that

$$f(P_1) \subset X \text{ and } f(P_0) \subset A.$$

for $X := |\mathcal{X}|$ and $A := |\mathcal{A}|$. There are many choices for such sets but picking up an arbitrary pair of sets with the above property would very likely lead to a loss of information about the dynamics of f. Let us momentarily postpone the construction of the target spaces and assume that the combinatorial map $\mathcal{F} : (\mathcal{P}_1, \mathcal{P}_0) \rightrightarrows (\mathcal{X}, \mathcal{A})$ is given.

As was indicated earlier $\mathcal{F}$ gives rise to a lower semicontinuous cubical map $F : P_1 \rightrightarrows X$ given by $F = \lfloor \mathcal{F} \rfloor$. Moreover, since $\mathcal{F}$ is an enclosure of f, we have the inclusion and identity

$$f(x) \in F(Q) = |\mathcal{F}(Q)| \tag{10.31}$$

for any $x \in P_1$ and any $Q \in \mathcal{K}_{\max}(P_1)$ containing x. Thus F is a map on pairs $F : (P_1, P_0) \rightrightarrows (X, A)$.

Assume for the moment that the cubical enclosure $\mathcal{F}$ is such that the corresponding cubical map F is acyclic-valued. The discussion of conditions under which this assumption can be satisfied is postponed until the next subsection. Under our assumption, F is a lower semicontinuous cubical representation of f, so we have the map in relative homology

$$f_* = F_* : H_*(P_1, P_0) \to H_*(X, A). \tag{10.32}$$

If we think that f_* algebraically encodes the dynamics of f, then ideally, we would like to have a self-map on $H_*(P_1, P_0)$. This would be possible if (X, A) could be chosen in such a way that $H_*(X, A)$ is isomorphic to $H_*(P_1, P_0)$. The following lemma indicates that there is a systematic means of doing this.

Lemma 10.82 *Let (P_1, P_0) be a cubical index pair for f given by Theorem 10.79. Define*

$$X := P_1 \cup F(P_0) \text{ and } A := P_0 \cup F(P_0).$$

Then

(a) $(P_1, P_0) \subset (X, A)$ and $F(P_1, P_0) \subset (X, A)$;

(b) The inclusion map $\iota : (P_1, P_0) \to (X, A)$ *induces an isomorphism in relative homology*

$$\iota_* : H_*(P_1, P_0) \xrightarrow{\cong} H_*(X, A).$$

Proof. The first inclusion $(P_1, P_0) \subset (X, A)$ in (a) is obvious and also it is obvious that $F(P_0) \subset A$. We leave the reader to check that the formula (10.28) in the proof of Theorem 10.79 is equivalent to

$$\mathcal{F}(\mathcal{P}_1) \subset \mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_0)$$

and this in turn is equivalent to

$$F(P_1) \subset P_1 \cup F(P_0)$$

so the second inclusion in (a) holds true.

We prove (b) by applying the Axcision Isomorphism Theorem 9.12. Define $U := F(P_0) \setminus P_1$. We verify that U satisfies the hypothesis of that theorem.

Obviously $U \subset F(P_0) \subset A$. We next need to show that U is open in X. Since P_1 is a cubical set, it is compact and hence it is closed in any subspace of $\mathbf{R}^n$. Thus it is closed in X. Therefore $X \setminus P_1$ is open in X. But

$$X \setminus P_1 = (P_1 \cup F(P_0)) \setminus P_1 = F(P_0) \setminus P_1 = U$$

hence U is open in X. The conclusion will follow from Theorem 9.12 once we show that

$$X \setminus U = P_1 \text{ and } A \setminus U = P_0.$$

The first identity is a straightforward set algebra:

$$X \setminus U = (F(P_0) \cup P_1) \setminus (F(P_0) \setminus P_1) = P_1.$$

For the second identity, we first verify the inclusion

$$P_0 \subset A \setminus U.$$

Indeed, if $x \in P_0$, then $x \notin U$ because $P_0 \subset P_1$. But $P_0 \subset A$ so $x \in A \setminus U$. It remains to verify the inclusion

$$A \setminus U \subset P_0.$$

We argue by contradiction. Suppose that there exists $x \in A \setminus U$ such that $x \notin P_0$. Then $x \in F(P_0) \setminus (F(P_0) \setminus P_1) = F(P_0) \cap P_1$. It follows from the formula (10.30) that

$$F(P_0) \cap P_1 \subset P_0$$

which contradicts the hypothesis that $x \notin P_0$. $\square$

Now we are ready to give the following definition.

Definition 10.83 Let $P = (P_1, P_0)$ be a cubical index pair for f given by Theorem 10.79. The associated *index map* is defined by

$$f_{P*} := \iota_*^{-1} \circ f_* : H_*(P_1, P_0) \to H_*(P_1, P_0)$$

As the following theorem indicates the index map can be used to prove the existence of a nontrivial isolated invariant set. However, to state the theorem we need to introduce the following notion.

Definition 10.84 Let G be an abelian group. A homomorphism $L : G \to G$ is *nilpotent* if there exists a positive integer n such that $L^n = 0$.

Theorem 10.85 *Let $P = (P_1, P_0)$ be a cubical index pair for f given by Theorem 10.79. If the associated index map f_{P*} is not nilpotent, then* $\operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), f) \neq \emptyset$.

This theorem, which can be interpreted as a homological version of the Ważewski principle for maps, lies at the heart of the Conley index theory. Unfortunately, unlike the setting of flows where the proof follows from the construction of a deformation retraction, the proof of this theorem is not straightforward and we will return to it at the end of this section.

One of the themes that we have been pursuing is that we can use a combinatorial enclosure $\mathcal{F}$ and homology to make conclusions about the dynamics of a continuous map f. For example, assume that using $\mathcal{F}$ we obtain an index pair (P_1, P_0) and that the resulting index map f_{P*} is not nilpotent, then by Theorem 10.85,

$$S := \operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), f) \neq \emptyset.$$

Thus, we are able to use $\mathcal{F}$ to conclude the existence of a nontrivial invariant set. Of course, there is nothing unique about the choice of $\mathcal{F}$. So assume we use a different combinatorial enclosure $\mathcal{F}'$ and obtain a different index pair (Q_1, Q_0) for f, but that

$$S = \operatorname{Inv}(\operatorname{cl}(Q_1 \setminus Q_0), f).$$

Since the index pairs are different it is not at all obvious that f_{Q*} must also be nilpotent. The Conley index theory assures us that it is. More succinctly, it tells us that f_{P*} is nilpotent if and only if f_{Q*} is nilpotent. Thus we can apply Theorem 10.85 independently of the particular index pair used. Unfortunately, a complete discussion of the Conley index is far beyond the scope of this book, therefore we will only give a brief description and discuss some important consequences.

Consider two index pairs (P_1, P_0) and (Q_1, Q_0) for a continuous map f. The induced index maps are f_{P*} and f_{Q*}. If we add the additional assumption that

$$S = \operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), f) = \operatorname{Inv}(\operatorname{cl}(Q_1 \setminus Q_0), f).$$

then the Conley index theory guarantees that the two index maps are equivalent in the following sense.

Definition 10.86 Two group homomorphisms between abelian groups, $f : G \to G$ and $g : G' \to G'$, are *shift equivalent* if there exist group homomorphisms $r : G \to G'$ and $s : G' \to G$ and a natural number m such that

$$r \circ f = g \circ r, \quad s \circ g = f \circ s, \quad r \circ s = g^m \quad s \circ r = f^m.$$

It is left as an exercise to show that shift equivalence defines an equivalence relation.

The following proposition shows that shift equivalence preserves nilpotency.

Proposition 10.87 *Let $f : G \to G$ and $g : G' \to G'$ be group homomorphisms that are shift equivalent. Then f is nilpotent if and only if g is nilpotent.*

Proof. Observe that it is sufficient to prove that if f is not nilpotent then g is not nilpotent. Since f is not nilpotent and $s \circ r = f^m$, neither r nor s can be trivial maps. Assume that g is nilpotent or more specifically that $g^k = 0$. Then,

$$\begin{aligned}
r \circ f &= g \circ r \\
s \circ r \circ f \circ (s \circ r)^k &= s \circ g \circ r \circ (s \circ r)^k \\
f^m \circ f \circ (f^m)^k &= s \circ g \circ (r \circ s)^k \circ r \\
f^{m(k+1)+1} &= s \circ g \circ (g^m)^k \circ r \\
f^{m(k+1)+1} &= 0.
\end{aligned}$$

This contradicts the assumption that f is not nilpotent. $\square$

As the following example shows shift equivalence is a stronger relation than nilpotency.

Example 10.88 Let $f, g : \mathbf{Z} \to \mathbf{Z}$ be the group homomorphisms given by

$$f(x) = 2x \quad \text{and} \quad g(x) = x.$$

Clearly, both f and g are not nilpotent. However as the following argument shows they are not shift equivalent. If they were, then there would exist group homomorphisms $r, s : \mathbf{Z} \to \mathbf{Z}$ such that

$$r \circ f = g \circ r \quad \text{and} \quad r \circ s = f^m$$

for some positive integer m. The first equation can only be solved by setting $r = 0$ which means that the second equation has no solution.

Shift equivalence can be used to define a form of the Conley index.

Definition 10.89 Let $f : X \to X$ be a continuous map. Let S be an isolated invariant set and let (P_1, P_0) be an index pair for f such that $S = \operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), f)$. The *homology Conley index* for S is the shift equivalence class of the index map f_{P*}.

The deep result, which we shall not prove, is that the Conley index of an isolated invariant set S is well defined. In other words, up to shift equivalence the index map does not depend on the index pair chosen. We conclude this section by showing that the Conley index implies Theorem 10.85.

Proof of Theorem 10.85: We shall prove the contrapositive; if $\operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), f) = \emptyset$, then f_{P*} is nilpotent.

Recall Example 10.71 where it was demonstrated that $\emptyset$ is an isolated invariant set for any continuous function f. Observe that $(\emptyset, \emptyset)$ is an index pair for $\emptyset$. Let $g_* : H_*(\emptyset, \emptyset) \to H_*(\emptyset, \emptyset)$ be the induced index map. Clearly, $g_* = 0$ and hence g_* is nilpotent.

Since the Conley index is well defined, g_* and f_{P*} are shift equivalent. By Example 10.88, the fact that g_* is nilpotent implies that f_{P*} is nilpotent. $\square$

Exercises

10.19 Prove that shift equivalence defines an equivalence relation on the set of group homomorphisms between abelian groups.

10.6.2 Topological Conjugacy

Up to this point we have concentrated on finding relatively simple dynamical structures such as fixed points and periodic orbits. However, nonlinear systems can exhibit extremely complicated structures often referred to as chaotic dynamics or simply chaos. While there is no universally accepted definition of chaos, there are particular examples that everyone agrees are chaotic. The best understood are those which can be represented in terms of symbolic dynamics. In this section we shall indicate how homology can be used to verify the existence of this type of chaos, but for the moment we need to deal with the issues of what does it mean to represent one dynamical system in terms of another. Let $f : X \to X$ and $g : Y \to Y$ be continuous maps. The following definition provides a topological means of concluding that the two dynamical systems they generate are equivalent.

Definition 10.90 Let $f : X \to X$ and $g : Y \to Y$ be continuous maps. A homeomorphism $\rho : X \to Y$ is a *topological conjugacy* from f to g if $\rho \circ f = g \circ \rho$ or equivalently if

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \rho & & \downarrow \rho \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes. We can weaken the relationship between f and g by only assuming that ρ is a continuous surjective map. In this case it is called a *topological semiconjugacy*.

Observe that if $\gamma_x : \mathbf{Z} \to X$ is a solution of f through x then $\sigma_{\rho(x)} : \mathbf{Z} \to Y$ given by

$$\sigma_{\rho(x)} := \rho \circ \gamma_x$$

is a solution of g through $\rho(x)$. If ρ is a homeomorphism, then ρ^{-1} exists and hence given a solution $\gamma_y : \mathbf{Z} \to Y$ of g through y we can define a solution $\sigma_x := \rho^{-1} \circ \gamma_y$ of f through $x := \rho^{-1}(y)$. Therefore, if two dynamical systems are related by a topological conjugacy, then there is an exact correspondence between all the orbits of each system.

As a first application of topological conjugacy, let us consider the question of how much information a particular combinatorial enclosure $\mathcal{F}$ can provide about the continuous map f. As the following example illustrates, there is no guarantee that $\mathcal{F}$ provides us with any knowledge about the dynamics of f.

Example 10.91 Consider the family of logistic maps $f_\lambda : [0,1] \to [0,1]$ given by $f_\lambda(x) = \lambda x(1-x)$. For all $\lambda \in [0,4]$ a cubical enclosure is given by the unique map $\mathcal{F} : \mathcal{K}_{\max}([0,1]) \to \mathcal{K}_{\max}([0,1])$, which sends the elementary interval $[0,1]$ to itself. Observe that $f_0(x) = 0$. Thus, the homological information that we can extract from $\mathcal{F}$ cannot indicate any dynamics more complicated than the constant map f_0. However, as we shall see in Example 10.93 it can be shown that f_4 generates many interesting orbits.

A possible remedy to the problem described in the above example is to pass to smaller units, that is to rescale the space. The decision on how large the rescaling should be can only be made experimentally. However, it should be clear that the larger the scaling the easier it should be to separate individual orbits. The following example indicates that dynamics generated by the rescaled system is conjugate to the dynamics of the original system.

Example 10.92 Let X be a cubical set and $f : X \to X$ a continuous map. Recall that rescaling $\Lambda^\alpha : X \to X^\alpha$ is a linear map and a homeomorphism with the inverse $\Omega^\alpha : X^\alpha \to X$. Recall that $f^\alpha : X^\alpha \to X$ has been defined as $X^\alpha = f \circ \Omega^\alpha$. Define

$$\tilde{f^\alpha} := \Lambda^\alpha \circ f^\alpha : X^\alpha \to X^\alpha.$$

It is instantly verified that Λ^α is a conjugacy from f to $\tilde{f^\alpha}$.

Topological conjugacy also plays an important role in justifying the assumption of the previous subsection that $F = \lfloor \mathcal{F} \rfloor$ is acyclic valued.

In Chapter 6 we used rescaling in the domain space X of a continuous map $f : X \to Y$, where X and Y are given cubical sets, in order to obtain an acyclic-valued representation and to define the map $f_* : H_*(X) \to H_*(Y)$. When constructing the index map, we search for an enclosure $\mathcal{F} : \mathbf{R}^d \rightrightarrows \mathbf{R}^d$ of f and fix the domain and target space a posteriori. This gives us more freedom than if we had to construct an acyclic-valued representation with values in a particular space and allows us to assure the acyclicity without any further rescaling. For example, we could possibly construct $\mathcal{F}$ with rectangular values which implies that the values of $F = \lfloor \mathcal{F} \rfloor$ are rectangular too (see [3] for a more detailed discussion along those lines). In this approach the rescaling is only needed to assure that the values of $\mathcal{F}$ are small enough to extract interesting dynamics. Considering maps with rectangular values is a sure but not the most economic way of approximating continuous maps. Therefore more sophisticated algorithms are used and the verification that a constructed map is acyclic-valued is a part of the algorithm.

We are finally ready to turn our attention to topological conjugacy and complicated dynamics. The following example involving the logistic map is useful starting point.

Example 10.93 Let $f : [0,1] \to [0,1]$ be given by $f(x) = 4x(1-x)$. Let $g : [0,1] \to [0,1]$ be given by

$$g(y) = \begin{cases} 2y & \text{if } 0 \le y \le 0.5, \\ 2(1-y) & \text{if } 0.5 < y \le 1. \end{cases}$$

Simple calculations show that the homeomorphism $\rho : [0,1] \to [0,1]$ given by

$$\rho(x) = \frac{2}{\pi} \arcsin \sqrt{x}$$

is a topological conjugacy, that is $\rho(f(x)) = g(\rho(x))$ for every $x \in [0,1]$.

In Example 10.93 g is a piecewise linear map and hence it is relatively easy to study individual trajectories. Since we have a topological conjugacy any trajectory of g corresponds to a trajectory in f and vice versa, thus we can use the dynamics of g to describe the dynamics of f. For example, it is fairly easy to discover that $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ is a period 2 orbit for g. Therefore, f contains a period two orbit. Of course, even though g is a reasonably simple map, a complete description of all its orbits is not obvious. For example, answering the question of whether or not g has a periodic orbit of period 17 requires some rather tedious work. With this in mind we would like to have a family of interesting but well understood dynamical systems. This was one of the reasons for introducing symbolic dynamics in Section 10.1.

What may appear puzzling at the moment is how one can relate the dynamics of a continuous map $f : X \to X$ to a particular subshift $\sigma_A : \Sigma_A \to \Sigma_A$. To avoid technical complications, let us assume that f is a homeomorphism. Now consider a collection of closed mutually disjoint sets $N_1, \ldots, N_n \subset X$. Let

$$S = \operatorname{Inv}\left(\bigcup_{i=1}^{n} N_i, f\right).$$

Our goal is to understand the structure of the dynamics of S. Since S is an invariant set $f(S) = S$. In particular, if $x \in S$, then

$$f^k(x) \in S \subset \bigcup_{i=1}^{n} N_i$$

for every $k \in \mathbf{Z}$. Furthermore, since the N_i are mutually disjoint, for each k there exists a unique i such that $f^k(x) \in N_i$. Define $\rho : S \to \Sigma_n$ by

$$\rho(x)_k = i \quad \text{if } f^k(x) \in N_i. \tag{10.33}$$

Since the N_i are disjoint closed sets, $\rho : S \to \Sigma_n$ is continuous. The reader should check that the following is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ \big\downarrow \rho & & \big\downarrow \rho \\ \Sigma_n & \xrightarrow{\sigma} & \Sigma_n \end{array} \tag{10.34}$$

Notice, however, that while this diagram is correct, it does not provide us with any information. Consider for example, the extreme case where S consists of a single fixed point. Then $\rho(S)$ is a single point and so there is nothing to be gained by knowing that $\rho : S \to \Sigma_n$.

On the other hand, assume for the moment that we could find a transition matrix A and an invariant set $S' \subset S$ such that

$$\begin{array}{ccc} S' & \xrightarrow{f} & S' \\ \big\downarrow \rho & & \big\downarrow \rho \\ \Sigma_A & \xrightarrow{\sigma_A} & \Sigma_A \end{array} \tag{10.35}$$

commuted and $\rho(S') = \Sigma_A$. In this case we could conclude that for every orbit in Σ_A there was a similar orbit in S. To be more precise, assume, for example, that the periodic point $(\overline{1,2}) \in \Sigma_A$, then we could conclude that there is an orbit

$$\{\ldots, x_{-1}, x_0, x_1, \ldots\} \subset S$$

with the property that $x_i \in N_1$ if i is even and $x_i \in N_2$ if i is odd.

Of course, since we have not assumed that ρ is a homeomorphism, we cannot claim that there is a unique orbit in S with this property. Thus, this approach of finding a transition matrix A such that $\rho : S' \to \Sigma_A$ is a surjective map, i.e. producing a topological semiconjugacy, can only provide a lower bound on the complexity of the dynamics in S. On the other hand, as we shall now indicate homology can be used to determine A.

Exercises

10.20 Prove that ρ in Example 10.93 defines a topological conjugacy.

10.21 Prove that $\rho : S \to \Sigma_n$ defined by (10.33) is continuous and that the diagram (10.34) commutes.

10.7 Computing Chaotic Dynamics

11

Homology of Topological Polyhedra

This chapter is primarily of interest to those of you who have some minimum knowledge of simplicial homology theory, at least, who know what a triangulation is, and would like to know how is our cubical approach related to the classical simplicial theory.

If you have no idea of what the simplicial theory is, you will find the definitions of the most basic concepts, results, and examples here but this exposition will be too brief to learn the whole theory from it. In particular we shall only comment but not present the concepts of barycentric subdivisions and simplicial approximations. The proofs of the presented results and more examples may be found in standard textbooks on Algebraic Topology, e.g.[32, 38].

11.1 Simplicial Homology

The cubical theory that has been developed throughout most of this chapter was built upon on the geometry of cubes in $\mathbf{R}^d$. The simplicial theory which is by far the most common is based on simplicies. Therefore, to present it we need to begin with some geometric preliminaries.

A subset K of $\mathbf{R}^d$ is called *convex* if, given any two points $x, y \in K$, the line segment

$$[x, y] := \{\lambda x + (1-\lambda)y \mid 0 \leq \lambda \leq 1\}$$

is contained in K.

Definition 11.1 The *convex hull* $\operatorname{co}A$ of a subset A of $\mathbf{R}^d$ is the intersection of all closed and convex sets containing A.

There is at least one closed convex set containing K, the whole space $\mathbf{R}^d$, hence $\operatorname{co}A \neq \emptyset$. It is easy to see that an intersection of any family of convex sets is convex and we already know that the same is true about intersections of closed sets. Thus $\operatorname{co}A$ is the smallest closed convex set containing A. It is intuitively clear that the convex hull of two points is a line segment joining those points, a convex hull of three non-co-linear points is a triangle, and a convex hull of four non-coplanar points is a tetrahedron. We shall generalize those geometric figures to an arbitrary dimension under the name simplex.

Theorem 11.2 *Let $\mathcal{V} = \{v_0, v_1, \ldots, v_n\} \in \mathbf{R}^d$ be a finite set. Then co$\mathcal{V}$ is the set of those $x \in \mathbf{R}^d$ which can be written as*

$$x = \sum_{i=0}^{n} \lambda_i v_i,\ 0 \leq \lambda_i \leq 1, ; \sum_{i=0}^{n} \lambda_i = 1\ . \tag{11.1}$$

In general, the coefficients λ_i need not be unique. For example, consider $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$ and $v_4 = (0,1)$, the four vertices of the unit square on Figure 2.5. Observe that,

$$(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}v_1 + 0v_2 + \frac{1}{2}v_3 + 0v_4 = 0v_1 + \frac{1}{2}v_2 + 0v_3 + \frac{1}{2}v_4\ .$$

Definition 11.3 A finite set $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$ in $\mathbf{R}^d$ is *geometrically independent* if, for any $x \in \mathrm{co}\mathcal{V}$, the coefficients λ_i in (11.1) are unique. If this is the case, λ_i are called *barycentric coordinates* of x.

The proof of the following result is left as an exercise.

Theorem 11.4 *Let $\mathcal{V} = \{v_0, v_1, \ldots, v_n\} \in \mathbf{R}^d$. Then $\mathcal{V}$ is geometrically independent if and only if the set of vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ is linearly independent. When this is the case, the barycentric coordinates of $x \in \mathcal{V}$ are continuous functions of x.*

Definition 11.5 Let $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$ be geometrically independent. The set $S = \mathrm{co}\mathcal{V}$ is called *simplex* or, more specifically, an *n-simplex spanned by vertices* $v_0, v_1, \ldots, v_n$. In certain situations we might want to use the term *geometric simplex* for S in order to distinguish it from an abstract oriented simplex which will be defined later. The number n is called *the dimension of* S. If $\mathcal{V}'$ is a subset of $\mathcal{V}$ of $k \leq n$ vertices, the set $S' = \mathrm{co}\mathcal{V}'$ is called *k-face* of S.

From Theorem 11.4 we get the following

Corollary 11.6 *Any two n-simplices are homeomorphic.*

Proof. Let $S = \mathrm{co}\{v_0, v_1, \ldots, v_n\}$ and $T = \mathrm{co}\{w_0, w_1, \ldots, w_n\}$ be two n-simplices. Let $\lambda_i(x)$ be barycentric coordinates of $x \in S$ and $\lambda_i(y)$ barycentric coordinates of $y \in T$. By the definition of geometric independence and by Theorem 11.4 the formula

$$f(x) := \sum_{i=0}^{n} \lambda_i(x) w_i$$

defines a linear continuous map $f : S \to T$ with the continuous inverse

$$f^{-1}(y) := \sum_{i=0}^{n} \lambda_i(y) v_i \ .$$

□

Definition 11.7 A *simplicial complex* $\mathcal{S}$ is a finite collection of simplices such that

1. Every face of a simplex in $\mathcal{S}$ is in $\mathcal{S}$;
2. The intersection of any two simplices in $\mathcal{S}$ is a face of each of them.

Definition 11.8 Given a simplicial complex $\mathcal{S}$ in $\mathbf{R}^d$, the union of all simplices of $\mathcal{S}$ is called the *polytope* of $\mathcal{S}$ and is denoted by $|\mathcal{S}|$. A subset $P \subset \mathbf{R}^d$ is a *polyhedron* if P is the polytope of some simplicial complex $\mathcal{S}$. In this case $\mathcal{S}$ is called a *triangulation* of P.

A polyhedron may have different triangulations. The Figure 11.1 shows examples of subdivisions of a square to triangles. The first two are triangulations but the last one is not since the intersection of a triangle in the lower-left corner with the triangle in the upper-right corner is not an edge of the latter one but a part of it. Observe that this is different from the cubical theory where a cubical subset of $\mathbf{R}^d$ has a unique cubical complex associated to it. Furthermore, any cubical set can be triangulated (see Exercise 11.6). However, as was shown in Exercise 2.6 not every polyhedron is cubical. This means that the simplicial theory is more flexible.

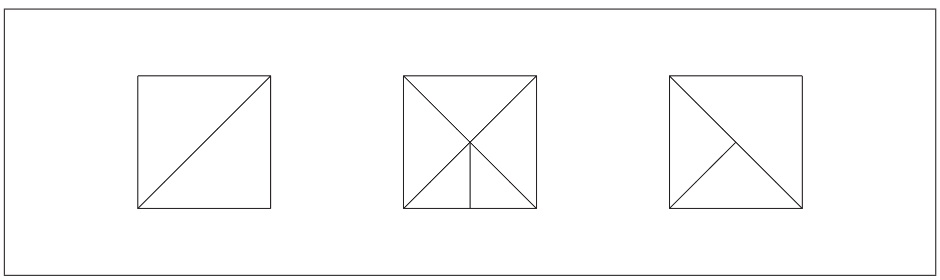

Fig. 11.1. Subdivisions of a square to triangles: the first two are triangulations, the last one is not.

Example 11.9 By a *torus* we mean any space homeomorphic to the product $S^1 \times S^1$ of two circles. Since $S^1 \times S^1 \in \mathbf{R}^4$, it is hard to visualize it. However one can show, by means of polar coordinates, that this space is homeomorphic to the surface in $\mathbf{R}^3$ obtained by rotation of the circle $(x-2)^2 + z^2 = 1,\ y = 0$ about the Y-axis. This set can be described as the surface of a donut. Neither of the above surfaces is a polyhedron but we shall construct one which is. Let G be the boundary of any triangle in $\mathbf{R}^2$. Then G is a simple closed curve hence it is homeomorphic to the unit circle. Thus $T = G \times G \in \mathbf{R}^4$ is a torus. In order to construct a triangulation of T we may visualize T as a square on Figure 11.2 with pairs of parallel sides glued together. More precisely, consider the square $[0,3]^2 = \mathrm{co}\{v_1, v_2, v_3, v_4\}$ where $v_1 = (0,0), v_2 = (0,3), v_3 = (3,3), v_4 = (0,3)$. Bend the square along the lines $x = 1$ and $x = 2$ and glue the directed edge $[v_1, v_4]$ with $[v_2, v_3]$ so that the vertex v_1 is identified with v_2 and v_4 with v_3. We obtain a cylinder in $\mathbf{R}^3$ with a boundary of a unilateral triangle in the plane $y = 0$ as the base. We bend the cylinder along the lines $y = 1$ and $y = 2$ (this cannot be done in $\mathbf{R}^3$ without stretching but we may add another axis) and glue the edge $[v_1, v_2]$ with $[v_4, v_3]$. Note that the four vertices v_1, v_2, v_3, v_4 of the square became one. The bend lines divide the square to nine unitary squares. Each of them can be divided to two triangles as shown on Figure 11.2. Let $\mathcal{S}$ be the collection of all vertices, edges, and triangles of T obtained in this way. Although some vertices and edges are identified by gluing, the reader may verify that $\mathcal{S}$ satisfies the definition of simplicial complex.

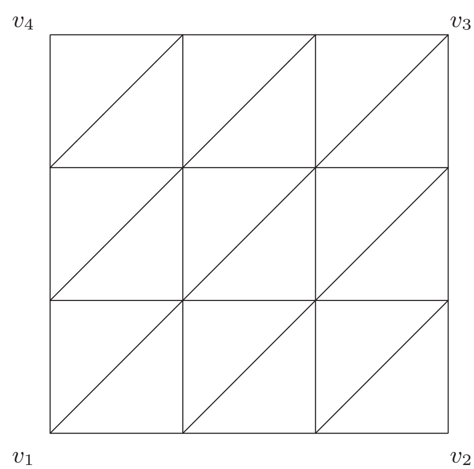

Fig. 11.2. Triangulation of a torus

Having finished with the geometric preliminaries we now turn to the problem of developing the associated algebra. The term simplicial complex suggests that there should be some natural structure of chain complex associated with it. That is not so easy to define due to problems with orientation which do not appear when we study cubical sets. We shall therefore proceed as we did with graphs in Chapter 1, that is, we shall start from chain complexes with coefficients in $\mathbf{Z}_2$. This will make definitions much more simple and, historically, this is the way homology was first introduced.

Let $C^n(\mathcal{S}; \mathbf{Z}_2)$ be the vector space generated by the set $\mathcal{S}^n$ of n-dimensional simplices of $\mathcal{S}$ as the canonical basis. More precisely, the basic vectors are duals $\widehat{S}$ of simplices S defined as in Definition 13.67. We put $C^n(\mathcal{S}; \mathbf{Z}_2) := 0$ if $\mathcal{S}$ has no simplices of dimension n. The boundary map $\partial_n : C^n(\mathcal{S}; \mathbf{Z}_2) \to C^{n-1}(\mathcal{S}; \mathbf{Z}_2)$ is defined on the dual of any basic element $S = \mathrm{co}\{v_0, v_1, \ldots, v_n\}$ by the formula

$$\partial_n(\widehat{S}) = \sum_{i=0}^{n} \mathrm{co}(\widehat{\mathcal{V} \setminus \{v_i\}}) \ .$$

Thus, in modulo 2 case, the algebraic boundary of a simplex corresponds precisely to its geometric boundary. We have the following

Proposition 11.10 $\partial_{n-1}\partial_n = 0$ *for all* n.

Proof. For any simplex $S = \mathrm{co}\{v_0, v_1, \ldots, v_n\}$,

$$\partial_{n-1}\partial_n(\widehat{S}) = \sum_{j \neq i} \sum_{i=0}^{n} \mathrm{co}(\widehat{\mathcal{V} \setminus \{v_i, v_j\}}) \ .$$

Each $(n-1)$-face of S appears in the above sum twice, therefore the sum modulo 2 is equal to zero. □

Thus $\mathcal{C}(\mathcal{S}; \mathbf{Z}_2) := \{C^n(\mathcal{S}; \mathbf{Z}_2), \partial_n\}_{n \in \mathbf{Z}}$ has the structure of a chain complex with coefficients in $\mathbf{Z}_2$. The homology of that chain complex is the sequence of vector spaces

$$H_*(\mathcal{S}; \mathbf{Z}_2) = \{H_n(\mathcal{S}; Z_2)\} = \{\ker \partial_n / \mathrm{im}\, \partial_{n+1}\}$$

The modulo 2 homology of graphs discussed in Section 1.4 is a special case of what we did above. The real goal however is to construct a chain complex corresponding to $\mathcal{S}$ with coefficients in $\mathbf{Z}$ as defined in Chapter 2. As we did it with graphs, we want to impose an orientation of vertices $v_0, v_1, \ldots, v_n$ spanning a simplex. In case of graphs that was easy since each edge joining vertices v_1, v_2 could be written in two ways, as $[v_1, v_2]$ or $[v_2, v_1]$ and it was sufficient to tell which vertex do we want to write as the first and which as the last. In case of simplices of dimension higher than one, there are many different ways of ordering the set of vertices.

Definition 11.11 Two orderings $(v_0, v_1, \ldots, v_n)$ and $(v_{p_0}, v_{p_1}, \ldots, v_{p_n})$ of vertices of an n-simplex S are said to have the same *orientation* if one can get one from another by an even number of permutations of neighboring terms

$$(v_{i-1}, v_i) \to (v_i, v_{i-1}) \ .$$

This defines an equivalence relation on the set of all orderings of vertices of S. An *oriented simplex* $\sigma = [v_0, v_1, \ldots, v_n]$ is an equivalence class of the ordering

$(v_0, v_1, \ldots, v_n)$ of vertices of a simplex $S = \mathrm{co}\{v_0, v_1, \ldots, v_n\}$. If S and T are geometric simplices, the corresponding oriented simplices are denoted by σ and, respectively, τ.

It is easy to see that for $n > 0$ the above equivalence relation divides the set of all orderings to two equivalence classes. Hence we may say that the orderings which are not in the same equivalence class have the *opposite orientation*. We shall denote the pairs or opposite oriented simplices by σ, σ' or τ, τ'. An oriented simplicial complex in a simplicial complex $\mathcal{S}$ with one of the two equivalence classes chosen for each simplex of $\mathcal{S}$. The orientations of a simplex and its faces may be done arbitrarily, they do not need to be related.

Example 11.12 Let S be a triangle in $\mathbf{R}^2$ spanned by vertices v_1, v_2, v_3. Then the orientation equivalence class $\sigma = [v_1, v_2, v_3]$ contains the orderings (v_1, v_2, v_3), (v_2, v_3, v_1), (v_3, v_1, v_2) and the opposite orientation σ' contains (v_1, v_3, v_2), (v_2, v_1, v_3), (v_3, v_2, v_1). One may graphically distinguish the two orientations by tracing a closed path around the boundary of the triangle s following the order of vertices. The first equivalence class gives the counter-clockwise direction and the second one the clockwise direction. However, the meaning of clockwise or counterclockwise orientation is lost when we consider a triangle in a space of higher dimension. Let $\mathcal{S}$ be the complex consisting of s and all of its edges and vertices. Here are some among possible choices of orientations and their graphical representations on Figure 11.3:

1. $[v_1, v_2, v_3], [v_1, v_2], [v_2, v_3], [v_3, v_1]$
2. $[v_1, v_2, v_3], [v_1, v_2], [v_2, v_3], [v_1, v_3]$
3. $[v_1, v_3, v_2], [v_1, v_2], [v_2, v_3], [v_1, v_3]$

On the first sight second and third orientation seem wrong since the arrows on the edges of the triangle do not close a cycle but do not worry: when we get to algebra, the "wrong" direction of the arrows will be corrected by the minus sign in the formula for the boundary operator.

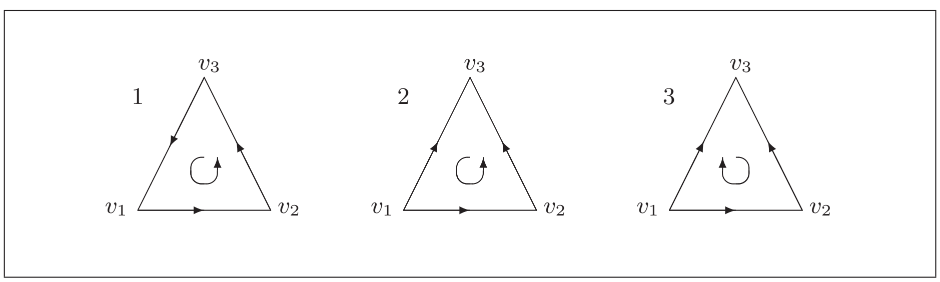

Fig. 11.3. Some orientations of simplices in a triangle

Let now S^n be the set of all oriented n-simplices of $\mathcal{S}$. Recall that a free abelian group $\mathbf{Z}^{S^n}$ generated by S^n is the set of all functions $c : S^n \to \mathbf{Z}$, generated by basic elements $\hat{\sigma}$ which can be identified with $\sigma \in S^n$. We would like to call this the group of n-chains but there is a complication: If $n \geq 0$, each n-simplex of $\mathcal{S}$ corresponds to two elements of S^n. We therefore adapt the following definition.

Definition 11.13 The group of *n-chains* denoted by $\mathcal{C}^n(\mathcal{S})$ is the subgroup of $\mathbf{Z}(S^n)$ consisting of those functions c which satisfy the identity

$$c(\sigma) = -c(\sigma')$$

if σ and σ' are opposite orientations of the same n-simplex s. We put $\mathcal{C}^n(\mathcal{S}) := 0$ if $\mathcal{S}$ contains no n-simplices.

Proposition 11.14 *The group $\mathcal{C}^n(\mathcal{S})$ is a free abelian group generated by functions $\tilde{\sigma} = \widehat{\sigma} - \widehat{\sigma'}$ given by the formula*

$$\tilde{\sigma}(\tau) := \begin{cases} 1 & \text{if } \tau = \sigma, \\ -1 & \text{if } \tau = \sigma', \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma, \sigma', \tau \in S^n$ and σ, σ' are opposite orientations of the same simplex. This set of generators is not a basis since $\tilde{\sigma'} = -\tilde{\sigma}$ for any pair σ, σ'. A basis is obtained by choosing either one.

The choice of a basis in Proposition 11.14 is related to the choice of an orientation in $\mathcal{S}$. Note that our notation of generators which correspond to geometric simplices became very complicated: We already have hats, primes, and now there comes tilde! We shall simplify this notation by identifying the basic elements $\tilde{\sigma}$ with σ. Upon this identification, we get the identification of σ' with $-\sigma$. The *simplicial boundary operator* $\partial_k : C^k(\mathcal{S}) \to C^{k-1}(\mathcal{S})$ is defined on any basic element $[v_0, v_1, \ldots, v_n]$ by the formula

$$\partial_k[v_0, v_1, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n] \; .$$

There is a bit of work involved in showing that this formula actually defines a boundary map: First, one needs to show that the formula is correct i.e. that it does not depend on the choice of a representative of the equivalence class $[v_0, v_1, \ldots, v_n]$. Secondly, one needs to show that $\partial_{k-1}\partial_k = 0$. The reader may consult [38] for the proofs.

Thus $\mathcal{C}(\mathcal{S}) := \{C^k(\mathcal{S}), \partial_k\}_{k \in \mathbf{Z}}$ has the structure of a chain complex as defined in Section **??**. The homology of that chain complex is the sequence of abelian groups.

$$H_*(\mathcal{S}) = \{H_n(\mathcal{S}\} = \{\ker \partial_n / \operatorname{im} \partial_{n+1}\} \; .$$

Exercises

11.1 Prove Theorem 11.4.

11.2 Define the chain complex $\mathcal{C}(T;\mathbf{Z}_2)$ for the triangulation discussed in Example 11.9 and use Homology program to compute $H_*(T;\mathbf{Z}_2)$.

11.3 Label vertices, edges, and triangles of the triangulation of the torus in Example 11.9 displayed on Figure 11.2. Define the chain complex $\mathcal{C}(T)$. Use Homology program to compute $H_*(T)$.

11.4 Let K be a polyhedron constructed as T in Example 11.9 but with one pair of sides twisted before gluing so that the directed edge $[a,d]$ is identified with $[c,b]$. The remaining pair of edges is glued as before, $[b,c]$ with $[a,d]$. Compute $H_*(K)$. What happens if we try to use Homology program for computing $H_*(K;\mathbf{Z}_2)$?
This K is called *Klein bottle*. Note that K cannot be visualized in $\mathbf{R}^3$, we need an extra dimension in order to glue two circles limiting a cylinder with twisting and without cutting the side surface of the cylinder.

11.5 Let P be a polyhedron constructed as T in Example 11.9 but with sides twisted before gluing so that the directed edge $[a,d]$ is identified with $[c,b]$ and $[b,c]$ with $[d,a]$. Compute $H_*(P)$. What happens if we try to use Homology program for computing $H_*(P;\mathbf{Z}_2)$?
This P is called *projective plane*. Note that P cannot be visualized in $\mathbf{R}^3$.

11.2 Comparison of Cubical and Simplicial Homology

Cubical complexes have several nice properties which are not shared by simplicial complexes:

1. As we already mentioned, numerical computations and computer graphics naturally lead to cubical sets. Since they already have a sufficient combinatorial structure to define homology, further subdivision to triangulations becomes artificial and increases the complexity of data. In particular, given a fixed embedding number d and the dimension n of an elementary cube P in a cubical set $X \subset \mathbf{R}^d$, we can give a precise upper bound for the number $\deg(P)$ of elementary cubes $Q \subset X$ of dimension $n+1$ which share P as a common face. That number was called in our algorithms in Chapter 4 *length of coboundaries* of P. For example, a vertex in a cubical set X in $\mathbf{R}^2$ is shared by at most four edges, in $\mathbf{R}^3$ by at most six edges. We leave as an exercise to estimate $\deg(P)$ in terms of d and n. The simplicial theory does not have this feature, moreover barycentric subdivisions increase the length of cobounaries.
2. A product of elementary cubes is an elementary cube but a product of simplices is not a simplex. For example, a product of a triangle by an interval is a cylinder and it has to be triangulated in order to compute the simplicial homology. That feature of elementary cubes makes many proves easier and lists of data shorter.

3. In Chapter 6 we used a concept of rescaling which replaced a traditional approach of cubical subdivisions. Rescaling formula is extremely simple. The notion of barycentric subdivision in the simplicial theory is much more complex both as a concept and as a numerical tool.
4. As we have seen in the previous section, the notion of orientation in simplicial complexes is not an easy concept to learn. Why does this problem not appear in the study of cubical complexes? The answer is in the fact that the definition of a cubical set is dependent on a particular choice of coordinates in the space. First, already in $\mathbf{R}$, we have unknowingly chosen a particular orientation by having written an elementary interval as $[l, l+1]$ and not $[l+1, l]$. In other words, a linear order of real numbers imposes a choice of an orientation on each coordinate axis in $\mathbf{R}^d$. Secondly, by having written a product of intervals $I_1 \times I_2 \times \cdots \times I_d$ we have implicitly chosen the ordering of the canonical basis for $\mathbf{R}^d$.

There is one weak point of cubical complexes: Every polyhedron can be triangulated but not every polyhedron can be expressed as a cubical set. In particular, a triangle is not a cubical set.

We have however the following theorem which will help us to define homology of a polyhedron via cubical homology. We start from a definition and observation.

Definition 11.15 Given any $d \geq 0$ the *standard d-simplex* Δ_d is given by $\Delta_d := \mathrm{co}\{e_1, e_2, \ldots e_{d+1}\}$ where $\{e_1, e_2, \ldots e_{d+1}\}$ is the canonical basis for $\mathbf{R}^{d+1}$.

It is easy to see that any linearly independent set is also geometrically independent so Δ_d is an d-simplex indeed. Its special property is that the barycentric coordinates of any point x in Δ_d coincide with the cartesian coordinates $x_1, x_2, \ldots x_{d+1}$.

Theorem 11.16 *Every polyhedron P is homeomorphic to a cubical set. Moreover, given any triangulation $\mathcal{S}$ of P, there exists a homeomorphism $h : P \to X$, where X is a cubical set, such that the restriction of h to any simplex of $\mathcal{S}$ is a homeomorphism of that simplex onto a cubical subset of X.*

Proof. In order to keep the idea transparent we skip several technical verifications. The construction of h is done in two steps.

Step 1. We construct a homeomorphic embedding of P into a standard simplex in a space of a sufficiently high dimension.

Indeed, let $\mathcal{S}$ be a triangulation of P and let $\mathcal{V} = \{v_1, v_2, \ldots, v_d\}$ be the set of all vertices of $\mathcal{S}$. Let Δ_d be the standard d-simplex in $\mathbf{R}^{d+1}$ described in Definition 11.15. Consider the bijection f_0 of $\mathcal{V}$ onto the canonical basis of $\mathbf{R}^{d+1}$ given by $f_0(v_i) = e_i$. Given any n-simplex $S = (co)\{v_{p_0}, v_{p_1} \ldots v_{p_n}\}$ of $\mathcal{S}$, f_0 extends to a map $f_S : S \to \mathbf{R}^{d+1}$ by the formula

$$f_S(\sum \lambda_i v_{p_i}) = \sum \lambda_i e_{p_i}$$

where λ_i are barycentric coordinates of a point in s. It follows that $f_S(S)$ is a n-simplex and f_S is a homeomorphism of S onto $f_S(S)$. If S and T are any two simplices of $\mathcal{S}$, $s \cap T$ is empty or is their common face so if $x \in s \cap T$ then

$$f_S(x) = f_{S \cap T}(x) = f_T(x) \; .$$

Thus the maps f_S match on intersections of simplices. Since simplices are closed and there are finitely many of them, the maps f_S extend to a map $f : P \to \tilde{P} := f(P)$. By the linear independence of $\{e_1, e_2, \ldots, e_d\}$, one shows that $\tilde{P}$ is a polyhedron triangulated by $\{f(S)\}$ and f is a homeomorphism. Moreover, by its construction, f maps simplices to simplices.

Step 2. We construct a homeomorphism g of Δ_d onto the cubical set $Y \subset \operatorname{bd}[0,1]^{d+1}$ consisting of those faces of $[0,1]^{d+1}$ which have the degenerate interval $[1]$ on one of the components and such that any face of Δ_d is mapped to a cubical face of Y. Once we do that, it will be sufficient to take $X := g(\tilde{P})$ and define the homeomorphism h as the composition of f and g.

Consider the diagonal line L parameterized by $t \to (t, t, \ldots, t) \in \mathbf{R}^{d+1}$, $t \in \mathbf{R}$. The idea is to project a point $x \in \Delta_d$ to a face of Y along the line L in the direction away from the origin. Recall that the barycentric coordinates of $x \in \Delta_d$ coincide with its cartesian coordinates, thus $\sum x_i = 1$ and $0 \leq x_i \leq 1$ for all i. The image $y = g(x)$ should have coordinates $y_i = x_i + t$ for all i and some $t \geq 0$. This point is in Y if $0 \leq x_i + t \leq 1$ for all i and $x_j + t = 1$ for some j. Note that the supremum norm of x is $||x|| = max\{x_1, x_2, \ldots x_{d+1}\}$. Thus the number $t := 1 - ||x||$ has the desired property and the coordinates of $y = g(x)$ are given by

$$y_i = 1 + x_i - ||x|| \; .$$

It is clear that g is continuous. The injectivity of g is proved by noticing that any line parallel to L intersects Δ_d at a unique point. The surjectivity of g is a by-product of the construction of its inverse g^{-1}. Let $y \in Y$. In order to define $x = g^{-1}(y)$ we must find a number $t \in [0,1]$ such that the point x whose coordinates are given by $x_i = y_i - t$ is in Δ_d. For this, we must have $0 \leq y_i - t \leq 1$ for all i and $\sum_{j=1}^{d+1} (y_j - t) = 1$. Thus

$$t = \frac{1}{d+1} (\sum_{j=1}^{d+1} y_j - 1) \; .$$

Since $0 \leq y_i \leq 1$ for all i and $y_j = 1$ for some j, t has the desired properties. □

Exercises

11.6 * Prove that any cubical set can be triangulated.

11.3 Homology Functor

11.3.1 Category of Cubical Sets

This and the following section requires some knowledge of basic definitions from the Category Theory. For a reader unfamiliar with this theory, we provide a brief summary of those definitions.

A category Cat is a pair (Obj, Mor) consisting of some collection of sets, say A, B, called *objects* and a collection of maps $f : A \to B$ from one object to another called *morphisms*. Given A, B in Obj, the collection of morphisms from A to B is denoted by $\mathrm{Cat}(\mathrm{A}, \mathrm{B})$. It is assumed that among those morphisms there must be the identity id_A on each object A and that a composition of a morphism in $\mathrm{Cat}(\mathrm{A}, \mathrm{B})$ with a morphism in $\mathrm{Cat}(\mathrm{B}, \mathrm{C})$ is a morphism in $\mathrm{Cat}(\mathrm{A}, \mathrm{C})$.

Here are some fundamental examples:

- The category Set has all sets as objects and all maps as morphisms.
- The category Top has all topological spaces as objects and all continuous maps as morphisms.
- The category Ab has all abelian groups as objects and all group homomorphisms as morphisms.
- The category Ab_* called *category of graded abelian groups* has all sequences $A_* = \{A_n\}_{n \in \mathbf{Z}}$ of abelian groups as objects and all sequences $\{f_n\}_{n \in \mathbf{Z}}$ of group homomorphisms $f_n : A_n \to B_n$ as morphisms.

There are "maps" $F : \mathrm{Cat} \to \mathrm{Cat}'$ from one category Cat to another Cat' called *functors* which permit comparing different categories and using the knowledge of one category to study another. Functors send objects A of Cat to objects $F(A)$ of Cat' and morphisms f of $\mathrm{Cat}(\mathrm{A}, \mathrm{B})$ to morphisms $F(f)$ of $\mathrm{Cat}'(\mathrm{F}(\mathrm{A}), \mathrm{F}(\mathrm{B}))$. Why do we call them functors and not just maps or functions? One of the problems is that if we did that, we would violate the Constitution - not the one of USA but the Constitution of Mathematics, that is, the list of primitive concepts and axioms which all mathematical theories must respect. Formally, maps and functions must be defined on sets but categories may be too "big" to be sets: the mentioned category Set is not a set since there exists no set of all sets!. Indeed, suppose that there exists a set S of all sets. Then the singleton $\{S\}$ also is a set but this violates the Constitution which says that a set cannot be an element of itself. Besides of that "pure formality", functors act basically like maps and they preserve compositions: A functor sends a composition of two morphisms in one category to a composition of images of those two morphims in another. More precisely, a *covariant functor* $F : \mathrm{Cat} \to \mathrm{Cat}'$ has the property

$$F(g \circ f) = F(g) \circ F(f),$$

whereas a *contravariant functor* $F : \mathrm{Cat} \to \mathrm{Cat}'$ has the property

$$F(g \circ f) = F(f) \circ F(g) \ .$$

The category of cubical spaces Cub is defined as follows. The objects in Cub are cubical sets. If X, Y are cubical sets then

$$\mathrm{Cub(X, Y)} := \{\mathrm{f : X \to Y} \mid \ f \text{ is continuous.}\}$$

Obviously Cub is a category.

The contents of Proposition 6.40 and Theorem 6.58 may be reformulated as the following theorem.

Theorem 11.17 $H_* : \mathrm{Cub} \to \mathrm{Ab}_*$ *is a covariant functor.*

11.3.2 Category of Topological Polyhedra

In Chapter 4 we saw that any polyhedron P is homeomorphic to a certain cubical set $h(P)$. Thus we may want to define the homology of P as the homology (up to isomorphism) of $h(P)$:

$$H_*(P) := H_*(h(P))$$

However, the construction of $h(P)$ in the Theorem 11.16 is not unique since it depends on a choice of triangulation. Is it unique up to isomorphism? Does so constructed homology have the functorial property we discussed in the previous section? In order to answer these question we shall use the language of the Category Theory, namely the concept of Connected Simple System. Some readers may find it too abstract but this actually is the shortest way to get a formal definition of the homology functor for that large class of spaces and the proof of its correctness.

A compact metric space K is called *topological polyhedron* or *representable space* if there exists a cubical set X and a homeomorphism $s : K \to X$. The collection of all topological polyhedra with all continuous maps forms a category which we shall denote by Pol. In particular, all geometric polyhedron is an object of Pol by Theorem 11.16. The triple (K, s, X) is called a *representation* of X.

As we said before, we want to define the homology of K by means of the homology of X.

11.3.3 Connected Simple Systems

Let Cat be a category and $\mathcal{E}$ its *small subcategory*, i.e. a category whose objects and morphisms are, respectively, objects and morphisms of Cat but the class of objects forms a set. We mentioned that the set of all sets does nor exist but, for example, the set $\mathcal{P}(\mathbf{Z})$ of all subsets of $\mathbf{Z}$ with all maps between them is a small category.

We will say that $\mathcal{E}$ is a *connected simple system* (CSS) in Cat if for any two objects $E_1, E_2 \in \mathcal{E}$ there exists exactly one morphism in $\mathcal{E}(E_1, E_2)$. If $\mathcal{E}$ is a connected simple system then the unique element of $\mathcal{E}(E_1, E_2)$ will be denoted by $\mathcal{E}_{E_1,E_2}$. Since $\mathcal{E}_{E_2,E_1} \circ \mathcal{E}_{E_1,E_2} = \mathcal{E}_{E_1,E_1}$ and the identity must be a morphism, it follows that all morphisms in $\mathcal{E}$ are isomorphisms. We may think about the set of all subsets of **Z** which have precisely n elements for a given n and all bijections between them.

We define the *category of connected simple systems* over Cat, denoted CSS (Cat), as follows. We take all connected simple systems in Cat as objects of this category. If $\mathcal{E}$ and $\mathcal{F}$ are two connected simple systems in Cat then by a morphism from $\mathcal{E}$ to $\mathcal{F}$ we mean any collection

$$\varphi := \{\varphi_{FE} \in \mathrm{Cat(E,F)} \mid \mathrm{E} \in \mathcal{E}, \mathrm{F} \in \mathcal{F}\}$$

of morphisms in Cat which satisfy

$$\varphi_{F'E'} = \mathcal{E}_{F'F}\varphi_{FE}\mathcal{E}_{EE'}$$

for any $E, E' \in \mathcal{E}, F, F' \in \mathcal{F}$. The elements of φ will be called *representants* of φ.

If $\psi := \{\psi_{GF} \in \mathrm{Cat(F,G)} \mid \mathrm{F} \in \mathcal{F}, \mathrm{G} \in \mathcal{G}\}$ is a morphism from $\mathcal{F}$ to $\mathcal{G} \in$ CSS (Cat) then it is straightforward to verify that for given objects $E \in \mathcal{E}, G \in \mathcal{G}$ the composition $\psi_{GF}\varphi_{FE}$ does not depend on the choice of an object $F \in \mathcal{F}$. Thus the morphism $(\psi\varphi)_{FE} := \psi_{GF}\varphi_{FE}$ is well defined and we can set

$$\psi\varphi := \{(\psi\varphi)_{GE} \mid E \in \mathcal{E}, G \in \mathcal{G}\}.$$

The commutativity of the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\varphi_{FE}} & F & \xrightarrow{\psi_{GF}} & G \\ \downarrow & & \downarrow & & \downarrow \\ E' & \xrightarrow{\varphi_{F'E'}} & F' & \xrightarrow{\psi_{G'F'}} & G' \end{array}$$

implies that $\psi\phi$ is a well defined composition of morphisms. It is now straightforward to verify that $\{\mathcal{E}_{E'E} \mid E, E' \in \mathrm{Obj}(\mathcal{E})\}$ is the identity morphism in $\mathcal{E}$. Thus we have proved

Theorem 11.18 CSS ($\mathcal{E}$) *is a category.*

The following proposition is straightforward.

Proposition 11.19 *Assume* $F : \mathrm{Cat} \to \mathrm{Cat}'$ *is a functor which maps distinct objects in* Cat *into distinct objects in* Cat′ *and* $\mathcal{E}$ *is a connected simple system in* Cat. *Then* $F(\mathcal{E})$ *is a connected simple system in* Cat′. *Moreover, if* $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ *is a morphism in* CSS (Cat) *then* $F(\varphi) : F(\mathcal{E}_1) \to F(\mathcal{E}_2)$ *given by*

$$F(\varphi) := \{F(\varphi_{E_1,E_2}) \mid E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\}$$

is a morphism in CSS (Cat′).

The above propositon lets us extend the functor $F : \mathrm{Cat} \to \mathrm{Cat}'$ to a functor $F : \mathrm{CSS\,(Cat)} \to \mathrm{CSS\,(Cat')}$.

11.3.4 Homology of Topological Polyhedra

We define Repr, the category of representations of compact metric spaces as follows. Its objects are all triples (K, s, X), where K is a compact space, X a cubical set and $s : K \to X$ a homeomorphism. If (K, s, X) and (L, t, Y) are two objects in Repr then the morphisms from (K, s, X) to (L, t, Y) are all continuous maps from X to Y. We extend the homology functor $H_* : \mathrm{Cub} \to \mathrm{Ab}_*$ to $H_* : \mathrm{Repr} \to \mathrm{Ab}_*$ by putting

$$H_*(K, s, X) := H_*(X) \times \{(K, s, X)\}.$$

By Proposition 11.19 $H_* : \mathrm{Repr} \to \mathrm{Ab}_*$ extends to the functor $H_* : \mathrm{CSS}\,(\mathrm{Repr}) \to \mathrm{CSS}\,(\mathrm{Ab}_*)$.

With every topological polyhedron K we associate a connected simple system $\mathrm{Rep}\,(K)$ in Repr defined as follows. The objects in $\mathrm{Rep}\,(K)$ are all representations of K. If (K, s, X) and (K, t, Y) are two representations of K then the unique morphism in $\mathrm{Rep}\,(K)((X, s), (Y, t))$ is the map ts^{-1}.

If K, L are two topological polyhedra and $f : K \to L$ is a continuous map then we define $\mathrm{Rep}\,(f) : \mathrm{Rep}\,(K) \to \mathrm{Rep}\,(L)$ as the collection of maps

$$\{tfs^{-1} \mid (X, s) \in \mathrm{Rep}\,(K), (Y, t) \in \mathrm{Rep}\,(L)\}.$$

One easily verifies that the above collection is a morphism in CSS (Repr).

We now define the homology functor $H_* : \mathrm{Pol} \to \mathrm{CSS}\,(\mathrm{Ab}_*)$ by

$$H_*(K) := H_*(\mathrm{Rep}\,(K))$$

and

$$H_*(f) := H_*(\mathrm{Rep}\,(f)).$$

There are many interesting questions to ask about the category of topological polyhedra and the cubical homology functor:

Is cubical homology equivalent to classical theories (simplicial, singular, and others) when restricted to the same class od spaces?

Are Eilenberg-Steenrod axioms (see [38] for example) for homology satisfied?

The answer to those question is beyond the scope of this book.

12

Topology

12.1 Norms and Metrics in $\mathbf{R}^d$

In order to study geometric properties of subsets of $\mathbf{R}^d$ and the behavior of maps from one set to another we need to know how to measure how far are two points $x, y \in \mathbf{R}^d$ from each other. In $\mathbf{R}$, the distance between two points is expressed by means of the absolute value of their difference $|x - y|$. In spaces of higher dimensions there are several natural ways of measuring the distance, each of them having some particular advantages. The most standard way is by means of the expression called *euclidian norm* which is given for each $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$ by

$$||x||_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}\ . \tag{12.1}$$

The number $||x||_2$ represents the length of the vector beginning at 0 and ending at x which we shall identify with the point x. This norm permits measuring the distance from x to y by the quantity

$$\operatorname{dist}_2(x, y) := ||x - y||_2\ . \tag{12.2}$$

The formula (12.1), however, becomes quite unpleasant in calculations since taking the square root is not a simple arithmetic operation. Thus, in numerical estimations, we often prefer to use the *supremum norm* of $x \in \mathbf{R}^d$ given by

$$||x||_0 := \sup_{1 \leq i \leq d} |x_i| = \max\{|x_1|, |x_2|, \ldots, |x_d|\}\ . \tag{12.3}$$

The related distance between points $x, y \in \mathbf{R}^d$ is

$$\operatorname{dist}_0(x, y) := ||x - y||_0\ . \tag{12.4}$$

Yet another convenient norm is the *sum norm* given by

$$||x||_1 := \sum_{i=1}^{d} |x_i| \tag{12.5}$$

with the related distance

$$\text{dist}_1(x, y) := ||x - y||_1 \ . \tag{12.6}$$

The indices appearing in the notation used for each consecutive norm and the related distance somewhat refer to exponents appearing in the sum defining them. Note that all three definitions of norms and distances coincide when the dimension is $d = 1$ and, in this case, we get the absolute value mentioned at the beginning.

Example 12.1 Consider points $x = (0, 0)$ and $y = (a, a)$ in $\mathbf{R}^2$ where $a > 0$. Then

$$\text{dist}_2(x, y) = a\sqrt{2}\ ,\ \text{dist}_0(x, y) = a\ ,\ \text{dist}_1(x, y) = 2a.$$

The notions which grasp all those cases are described in the following definitions.

Definition 12.2 A function $|| \cdot || : \mathbf{R}^d \to [0, \infty]$ is called *norm* if it has the following properties

(a) $||x|| = 0 \Leftrightarrow x = 0$ for all $x \in \mathbf{R}^d$ (*normalization*);
(b) $||cx|| = |c| \cdot ||x||$ for all $x \in \mathbf{R}^d$ and $c \in \mathbf{R}$ (*homogeneity*);
(c) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in \mathbf{R}^d$ (*triangle inequality*).

Definition 12.3 Let X be any set. A function $\text{dist} : X \times X \to [0, \infty]$ is called *metric* on X if it has the following properties

(a) $\text{dist}(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ (*normalization*);
(b) $\text{dist}(y, x) = \text{dist}(x, y)$ for all $x, y \in X$ (*symmetry*);
(c) $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$ for all $x, y, z \in X$ (*triangle inequality*).

A set X with a given metric dist is called *metric space*. More formally, a metric space is a pair (X, dist) which consists of a set and a metric on it.

Given any norm $|| \cdot ||$ in X, the formula

$$\text{dist}(x, y) = ||x - y||$$

defines a metric in X. The formulas (12.1), (12.3), (12.5) define norms in $\mathbf{R}^d$ and the formulas (12.2), (12.4), (12.6) define metrics in $\mathbf{R}^d$. We leave the proofs of those observations as exercises.

Since we will always work with subsets of $\mathbf{R}^d$, why do we bother about the definition of a metric and do not content ourselves just with the definition of a norm? The main reason is that norms are defined in linear spaces and subsets of $\mathbf{R}^d$ are not necessarily closed under addition and multiplication by scalars. This fact, however, does not affect the definition of a metric:

Proposition 12.4 *Let X be a metric space with a metric* dist *and let Y be any subset of X. Then the restriction of* dist *to $Y \times Y$ is a metric on Y.*

Proof. The conclusion is obvious. □

Example 12.5 Let $X = S_2^1 = \{x = (x_1, x_2) \in \mathbf{R}^d \mid x_1^2 + x_2^2 = 1\}$. S_2^1 is called *unit circle* in $\mathbf{R}^2$. The restriction of the euclidian distance to S_2^1 defines a metric on S_2^1 even if $x - y$ is not necessarily in S_2^1 for $x, y \in S_2^1$. The distance between points is the length of the line segment joining them in the outer space $\mathbf{R}^2$.
Another possible metric is obtained by defining $d(x, y)$ as the length of the shorter of the two arcs of S_2^1 joining x to y. Such an arc is called *geodesic*. Note that some pairs of points may have more then one geodesic, namely two half-circles when $y = -x$. When the space is a surface, the problem of finding geodesics is highly nontrivial: it is one of central problems of classical differential geometry.

Given a point x in a metric space (X, dist), we define the *ball* of radius r centered at x as

$$B(x, r) := \{y \in X \mid \mathrm{dist}(x, y) < r\}.$$

The set

$$S(x, r) := \{y \in X \mid \mathrm{dist}(x, y) = r\}$$

limiting the ball is called *sphere* of radius r centered at x. The balls in $\mathbf{R}^d$ for the metrics dist_0, dist_1, dist_2 are denoted, respectively, by $B_0(x, r)$, $B_1(x, r)$, $B_2(x, r)$, and the spheres by $S_0(x, r)$, $S_1(x, r)$, and $S_2(x, r)$. Figure 12.1 shows the unit spheres $S_2 := S_2(0, 1)$, $S_0 := S_0(0, 1)$ in $\mathbf{R}^2$ whereas sketching the sphere $S_1 := S_1(0, 1)$ is left as an exercise. When the dimension of a sphere or a ball is important but not clear from the context, we will add it as a superscript, e.g. B_0^d is a unit ball with respect to the supremum norm in $\mathbf{R}^d$ and S_0^{d-1} is the corresponding unit sphere in $\mathbf{R}^d$.

The central notion in topology is the notion of open set:

Definition 12.6 Let (X, dist) be a metric space. A set $U \subset X$ is *open* if and only if for every point $x \in U$ there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. The empty set $\emptyset$ is defined to be open.

Example 12.7 The interval $(-1, 2) \subset \mathbf{R}$ is an open set with respect to the standard norm $|x|$ on $\mathbf{R}$. To prove this let $x \in (-1, 2)$. This is equivalent to the conditions $-1 < x$ and $x < 2$. Choose $r_0 = (x + 1)/2$ and $r_1 = (2 - x)/2$. Then, both $r_0 > 0$ and $r_1 > 0$. Let $\epsilon = \min\{r_0, r_1\}$. One easily verifies that $B_2(x, \epsilon) \subset (-1, 2)$. Since this is true for any $x \in (-1, 2)$, the conclusion follows

Generalizing this argument leads to the following result. Its proof is left as an exercise.

Proposition 12.8 *Any interval of the form (a, b), (a, ∞) or $(-\infty, b)$ is open in $\mathbf{R}$.*

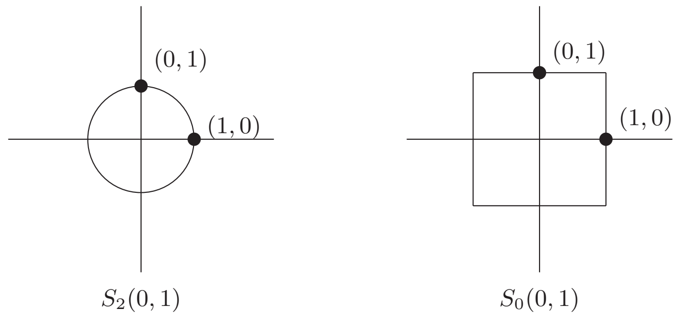

Fig. 12.1. The unit spheres for with respect to the euclidian and supremum norm.

From Definition 12.6, it follows that the arbitrary union of intervals is open, e.g. $(a,b) \cup (c,d)$ is an open set.

Example 12.9 The unit ball B_2 is an open set with respect to the metric dist $_2$. Observe that if $x \in B_2$ then $||x||_2 < 1$. Therefore, $0 < 1 - ||x||_2$. Let $r = \frac{1-||x||_2}{2}$. Then, $B_2(x,r) \subset B_2$.

Example 12.10 Of course not every set is open. As an example consider $(0,1] \subset \mathbf{R}$. By definition $1 \in (0,1]$ but, given any $\epsilon > 0$, $B_2(1,\epsilon) \not\subset (0,1]$. Therefore, $(0,1]$ is not open in $\mathbf{R}$. The same argument shows that any interval of the form $(a,b]$, $[a,b)$ or $[a,b]$ is not open in $\mathbf{R}$.

The following observation motivates the notion of topology introduced in the next section.

Proposition 12.11 *Let $U \subset \mathbf{R}^d$. The following conditions are equivalent*

(a) U is open with respect to the metric dist $_2$.
(b) U is open with respect to the metric dist $_0$.
(c) U is open with respect to the metric dist $_1$.

Proof. Let $U \subset \mathbf{R}^d$. Suppose that U is open with respect to dist $_0$ and let $x \in U$. Then there exists $\epsilon > 0$ such that $B_0(x,\epsilon) \subset U$. It is left as an exercise to check that

$$B_1(x,\epsilon) \subset B_2(x,\epsilon) \subset B_0(x,\epsilon) \subset U.$$

Since this is true for all $x \in U$, it follows that U is open with respect to dist $_2$, hence (b) implies (a). By the same arguments, (a) implies (c).

It is also easy to check that $B_0(x,\epsilon/2) \subset B_1(x,\epsilon)$ hence, by the same argument as above, (c) implies (b). Hence all three conditions are equivalent. □

In topological considerations we only care about open sets and not the exact metric. In the light of the above Proposition, we may choose anyone of the three discussed metrics which is most convenient and our choice is dist$_0$. For simplicity of notation, we shall abandon the index and write

$$||x|| := ||x||_0 \ , \ \text{dist}\,(x,y) := \text{dist}_0(x,y).$$

Exercises

12.1 Show that the functions defined by the formulas (12.3) and (12.5) are norms in the sense of Definition 12.2.

12.2 * Show that the function defined by the formula (12.1) is a norm in the sense of Definition 12.2.
Hint: For proving the triangle inequality find, in linear algebra textbooks, the proof of the *Cauchy-Schwartz inequality*:

$$|x \cdot y| \leq ||x||_2 \cdot ||y||_2$$

where

$$x \cdot y := \sum_{i=1}^{d} x_i y_i \ .$$

Observe that $||x||_2^2 = x \cdot x$. Deduce the triangle inequality from the Cauchy-Schwartz inequality.

12.3 Let $||\cdot||$ be a norm in $\mathbf{R}^d$ and $X \subset \mathbf{R}^d$. Show that the formula dist$\,(x,y) := ||x-y||$ for $x, y \in X$ defines a metric in X.

12.4 Sketch the set given by the equation $||x||_1 = 1$ in $\mathbf{R}^2$ as it is done for $||x||_2 = 1$ and $||x||_0 = 1$ in Figure 12.1.

12.5 Prove Proposition 12.8.

12.6 (a) Complete the proof of Proposition 12.11 by verifying that

$$B_1(x,\epsilon) \subset B_2(x,\epsilon) \subset B_0(x,\epsilon) \subset U.$$

(b) Show that

$$B_0(x, \frac{\epsilon}{\sqrt{d}}) \subset B_2(x,\epsilon)$$

for any $\epsilon > 0$ and use this fact to give a direct proof of the implication $(a) \Rightarrow (b)$ in Proposition 12.11.

12.2 Topology

As we saw in the previous section different metrics may give the same class of open sets. Since many concepts such as the continuity of functions may be entirely expressed in terms of open sets, it is natural to search for a definition of an open set which would not refer to a particular metric. This is done by listing axioms which the collection of all open sets should satisfy.

Definition 12.12 A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$.
3. Any *finite* intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

The elements of the topology $\mathcal{T}$ are called *open sets*. A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*.

This is a fairly abstract definition - fortunately we don't need to work at this level of generality. In fact in everything we do we will always assume that the set X is a subset of $\mathbf{R}^d$, so, by Proposition 12.4, it inherits the chosen metric d_0 from $\mathbf{R}^d$. Hence it is enough to know that the definition of an open set with respect to a given metric is consistent with the definition of a topology:

Theorem 12.13 *Let (X, d) be a metric space. Then the set of all subsets of X which are open in the sense of Definition 12.6 forms a topology on X.*

We leave the proof as an exercise.

In Proposition 12.11 we proved that all three discussed metrics induce the same topology on $\mathbf{R}^d$. It will be called the *standard topology*. Unless it is explicitly stated otherwise $\mathcal{T}$ will always be the standard topology.

As important as an open set is the notion of a closed set.

Definition 12.14 A subset K of a topological space X is *closed* if its complement

$$X \setminus K := \{x \in X \mid x \notin K\}$$

is open.

Example 12.15 The interval $[a, b]$ is a closed subset of $\mathbf{R}$. This is straightforward to see since its complement $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open. Similarly, $[a, \infty)$ and $(-\infty, b] \subset \mathbf{R}$ are closed.

Example 12.16 The set $\bar{B}_0 := \{x \in \mathbf{R}^d \mid ||x|| \leq 1\}$ is closed. This is equivalent to claiming that $\mathbf{R}^d \setminus \bar{B}_0$ is open, i.e. that $\{x \in \mathbf{R}^d \mid ||x|| > 1\}$ is open. Observe that $||x|| > 1$ is equivalent to $\max_{i=1,\ldots,d}\{|x_i|\} > 1$. Thus, there exists at least one coordinate, say the j-th coordinate, such that $|x_j| > 1$. Then

$$B_0(x, \frac{|x_j| - 1}{2}) \subset \mathbf{R}^d \setminus \bar{B}_0.$$

What we have proved justifies calling $\bar{B}_0$ the *closed unit ball* (with respect to the supremum norm). By the same arguments, the set

$$\bar{B}_0(x, r) := \{y \in \mathbf{R}^d \mid ||y - x|| \leq r\}$$

is closed is closed for every $x \in \mathbf{R}^d$ and every $r \geq 0$ so the name *closed ball* of radius r centered at x is again justified.

Remark 12.17 The reader should take care not to get lulled into the idea that a set is either open or closed. Many sets are *neither*. For example, the interval $(0, 1] \subset \mathbf{R}$ is neither open nor closed. As was observed in Example 12.10, it is not open. Similarly, it is not closed since its complement is $(-\infty, 0] \cup (1, \infty)$ not open.

Theorem 12.18 *Let X be a topological space. Then the following statements are true.*

1. *$\emptyset$ and X are closed sets.*
2. *Arbitrary intersections of closed sets are closed.*
3. *Finite unions of closed sets are closed.*

Proof. (1) $\emptyset = X \setminus X$ and $X = X \setminus \emptyset$.

(2) Let $\{K_\alpha\}_{\alpha \in \mathcal{A}}$ be an arbitrary collection of closed sets. Then

$$X \setminus \bigcap_{\alpha \in \mathcal{A}} K_\alpha = \bigcup_{\alpha \in \mathcal{A}} (X \setminus K_\alpha).$$

Since, by definition $X \setminus K_\alpha$ is open for each $\alpha \in \mathcal{A}$ and the arbitrary union of open sets is open, $X \setminus \bigcap_{\alpha \in \mathcal{A}} K_\alpha$ is open. Therefore, $\bigcap_{\alpha \in \mathcal{A}} K_\alpha$ is closed.

(3) See Exercise 12.9. □

Definition 12.19 Let X be a topological space and let $A \subset X$. The *closure* of A in X is the intersection of all closed sets in X containing A. The closure of A is denoted by $\operatorname{cl} A$ (many authors also use the notation $\bar{A}$.)

By Theorem 12.18 the arbitrary intersection of closed sets is closed, therefore the closure of an arbitrary set is a closed set.

Example 12.20 Consider $[0, 1) \subset \mathbf{R}$. Then $\operatorname{cl} [0, 1) = [0, 1]$. This is easy to show. First one needs to check that $[0, 1)$ is not closed. This follows from the fact that $[1, \infty)$ is not open. Then one shows that $[0, 1]$ is closed by showing that $(-\infty, 0) \cup (1, \infty)$ is an open set in $\mathbf{R}$. Finally one observes that any closed set that contains $[0, 1)$ must contain $[0, 1]$.

Similar argument shows that

$$\operatorname{cl} (0, 1) = \operatorname{cl} [0, 1) = \operatorname{cl} (0, 1] = \operatorname{cl} [0, 1] = [0, 1].$$

We leave as an exercise the proof of the following

Proposition 12.21 *Let $r > 0$. The closed ball $\bar{B}_0(x,r)$ defined in Example 12.16 is the closure of the ball $B_0(x,r)$. Explicitly,*

$$\bar{B}_0(x,r) = \operatorname{cl} B_0(x,r)$$

Up to this point, the only topological spaces that have been considered are those of $\mathbf{R}^d$ for different values of d. We saw in Proposition 12.4 that metrics can be restricted to subsets. Here is the analogy of that for topologies:

Definition 12.22 Let Z be a topological space with topology $\mathcal{T}$. Let $X \subset Z$. The *subspace topology* on X is the collection of sets

$$\mathcal{T}_X := \{X \cap U \mid U \in \mathcal{T}\}.$$

Before this definition can be accepted, the following proposition needs to be proved.

Proposition 12.23 *$\mathcal{T}_X$ defines a topology on X.*

Proof. The three conditions of Definition 12.12 need to be checked. First, observe that $\emptyset \in \mathcal{T}_X$ since $\emptyset = X \cap \emptyset$. Similarly, $X \in \mathcal{T}_X$ since $X = X \cap Z$.

The intersection and union properties follow from the following equalities:

$$\bigcap_{i=1}^{n} (X \cap U_i) = X \cap \left(\bigcap_{i=1}^{n} U_i \right)$$

$$\bigcup_{i \in \mathcal{I}} (X \cap U_i) = X \cap \left(\bigcup_{i \in \mathcal{I}} U_i \right)$$

□

Using this definition of the subspace topology, any set $X \subset \mathbf{R}^d$ can be treated as a topological space.

The following proposition is straightforward

Proposition 12.24 *If A is open in Z then $A \cap X$ is open in X. If A is closed in Z then $A \cap X$ is closed in X.*

It is important to notice that while open sets in the subspace topology are defined in terms of open sets in the ambient space, the sets themselves may "look" different.

Example 12.25 Consider the interval $[-1,1] \subset \mathbf{R}$ with the subspace topology induced by the standard topology on $\mathbf{R}$. $(0,2)$ is an open set in $\mathbf{R}$ hence

$$(0,1] = (0,2) \cap [-1,1]$$

is an open set in $[-1,1]$. We leave it to the reader to check that any interval of the form $[-1,a)$ and $(a,1]$ where $-1 < a < 1$ is an open set in $[-1,1]$.

Example 12.26 Let $X = [-1, 0) \cup (0, 1]$. Observe that $[-1, 0) = (-2, 0) \cap X$ and $(0, 1] = (0, 2) \cap X$, thus both are open sets. However, $[-1, 0) = X \setminus (0, 1]$ and $(0, 1] = X \setminus [-1, 0)$ so both are also closed sets. This shows that for general topological spaces one can have nontrivial sets that are both open and closed.

Given a pair $A \subset X$ we want to consider the set of points which separate A from its complement $X \setminus A$.

Definition 12.27 Let X be a topological space and $A \subset X$. The set

$$\operatorname{bd}_X A := \operatorname{cl} A \cap \operatorname{cl}(X \setminus A)$$

is called *topological boundary* or, shortly, *boundary* of A in X. If the space X is clear from the context, an abbreviated notation $\operatorname{bd} A$ may be used.

Example 12.28 Let $A := [-1, 1] \times \{0\}$ and $X = \mathbf{R} \times \{0\}$. Then

$$\operatorname{bd}_X A = \{(-1, 0), (1, 0)\}$$

which corresponds to the geometric intuition that a boundary of an interval should be the pair of its endpoints. However

$$\operatorname{bd}_{\mathbf{R}^2} A = A$$

because $\operatorname{cl}(\mathbf{R}^2 \setminus A) = \mathbf{R}^2$.

The above example shows that the concept of boundary of A is relative to the space in which A is sitting. Note also that

$$\operatorname{bd}_X X = \emptyset$$

for any space X because $X \setminus X = \emptyset$.

The proof of the following is left as an exception.

Proposition 12.29

$$\partial \bar{B}_0(x, r) = S_0(x, r)$$

Exercises

12.7 Prove that any set consisting of a single point is closed in $\mathbf{R}^d$.

12.8 Prove Theorem 12.13.

12.9 Prove that the intersection of finitely many closed sets is closed.

12.10 (a) Prove Proposition 12.21.
(b) Prove Proposition 12.29.

12.11 Let $Q = [k_1, k_1 + 1] \times [k_2, k_2 + 1] \times [k_3, k_3 + 1] \subset \mathbf{R}^3$ where $k_i \in \mathbf{Z}$ for $i = 1, 2, 3$. Prove that Q is a closed set.

12.12 Let $\overset{\circ}{Q} = \{k_1\} \times (k_2, k_2 + 1) \times (k_3, k_3 + 1) \subset \mathbf{R}^3$ where $k_i \in \mathbf{Z}$ for $i = 1, 2, 3$.

(a) Prove that $\overset{\circ}{Q}$ is neither open nor closed in $\mathbf{R}^3$.
(b) Show that $\overset{\circ}{Q}$ is open in the hyperplane $X = \{x \in \mathbf{R}^3 : x_1 = k_1\}$.
(c) Show that the closure of $\overset{\circ}{Q}$ in $\mathbf{R}^3$ is $\operatorname{cl} \overset{\circ}{Q} = \{k_1\} \times [k_2, k_2 + 1] \times [k_3, k_3 + 1]$.

12.13 Let Q be the *unit d-cube* in $\mathbf{R}^d$:

$$Q := \{x \in \mathbf{R}^d \mid 0 \leq x_i \leq 1\}.$$

Let

$$\overset{\circ}{Q} := \{x \in \mathbf{R}^d \mid 0 < x_i < 1\}$$

and

$$\Gamma^{d-1} := \operatorname{bd}_{\mathbf{R}^d} Q = \operatorname{cl} Q \cap \operatorname{cl} (\mathbf{R}^d \setminus Q).$$

Prove the following:

1. Q is closed in $\mathbf{R}^d$;
2. $\overset{\circ}{Q}$ is open in $\mathbf{R}^d$;
3. $\operatorname{cl} \overset{\circ}{Q} = Q$;
4. Γ^{d-1} is closed in $\mathbf{R}^d$;
5. $\Gamma^{d-1} = \{x \in Q \mid x_i \in \{0, 1\} \text{ for some } i = 1, \ldots, d\}$;
6. Γ^{d-1} is the union of proper faces of Q (see Definition 2.7);
7. Γ^{d-1} is the union of free faces of Q (see Definition 2.60);

12.14 * Let X be a cubical set in $\mathbf{R}^d$, X_1 the union of its maximal faces (see Definition 2.60) of dimension d, X_2 the union of its maximal faces of X of dimension less than d, and let $\texttt{Freeface}(X_1)$ be the union of all free faces of X_1. Show that

$$\operatorname{bd}_{\mathbf{R}^d} X = \texttt{Freeface}(X_1) \cup X_2$$

You may want to do Exercise 2.16 first.

12.3 Continuous Maps

As it was mentioned in the introduction, a frequently asked question is whether or not two given structures in $\mathbf{R}^d$ have a similar shape. Such comparisons can be done by means of continuous maps.

Example 12.30 The square $X := [0,1] \times [0,1] \subset \mathbf{R}^2$ and a portion of the closed unit disk $Y := \{x \in \mathbf{R}^2 \mid ||x||_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0\} \subset \mathbf{R}^2$ are clearly different from the geometric point of view: the first one is a polyhedron, the second one is not. However, we would like to think of them as being "equivalent" in a topological sense, since they can be transformed from one to the other and back by simply stretching or contracting the spaces.

To be more precise, observe that any element of Y has the form $y = (r\cos\theta, r\sin\theta)$ where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$. Define $f : Y \to X$ by

$$f(r\cos\theta, r\sin\theta) := \begin{cases} (r, r\tan\theta) & \text{if } 0 \leq \theta \leq \pi/4, \\ (r\cot\theta, r) & \text{if } \pi/4 \leq \theta \leq \pi/2. \end{cases}$$

It can be verified that f is well defined and continuous. Observe that this map just expands Y by moving points out along the rays emanating from the origin. One can also write down a map $g : X \to Y$ which shrinks X onto Y along the same rays (see Exercise 12.15).

Finally, one can construct a map from the square $\bar{B}_0^2 = [-1,1]^2$ onto the disc $\bar{B}_2^2 = \{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ by repeating the same construction for each quadrant of the plane. Note that either set represents a closed unit ball: the first one with respect to the supremum norm and the second one with respect to the euclidian norm (see Figure 12.1).

You have already seen maps of the form of f in the previous example in your calculus class under the label of a continuous functions. Since we introduced the notion of topology on an abstract level, we also want to define continuous functions in an equally abstract way.

Recall that a topological space consists of two objects, the set X and the topology $\mathcal{T}$. Therefore, to compare two different topological spaces one needs to make a comparison of both the elements of the sets - this is done using functions - and one needs to compare the open sets that make up the two topologies.

Definition 12.31 Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. A function $f : X \to Y$ is *continuous* if and only if for every open set $V \in \mathcal{T}_Y$ its preimage under f is open in X, i.e.

$$f^{-1}(V) \in \mathcal{T}_X.$$

Another useful way to characterize continuous functions is as follows.

Proposition 12.32 *Let $f : X \to Y$. f is continuous if and only if for every closed set $K \subset Y$, $f^{-1}(K)$ is a closed subset of X.*

Proof. ($\Rightarrow$) Let $K \subset Y$ be an a closed set. Then $Y \setminus K$ is an open set. Since f is continuous, $f^{-1}(Y \setminus K)$ is an open subset of X. Hence $X \setminus f^{-1}(Y \setminus K)$ is closed in X. Thus, it only needs to be shown that $X \setminus f^{-1}(Y \setminus K) = f^{-1}(K)$. Let $x \in X \setminus f^{-1}(Y \setminus K)$. Then $f(x) \in Y$ and $f(x) \notin Y \setminus K$. Therefore,

$f(x) \in K$ or equivalently $x \in f^{-1}(K)$. Thus, $X \setminus f^{-1}(Y \setminus K) \subset f^{-1}(X)$. Now assume $x \in f^{-1}(K)$. Then, $x \notin f^{-1}(Y \setminus K)$ and hence $x \in X \setminus f^{-1}(Y \setminus K)$.

($\Leftarrow$) Let $U \subset Y$ be an open set. Then $Y \setminus U$ is a closed subset. By hypothesis, $f^{-1}(Y \setminus U)$ is closed. Thus $X \setminus f^{-1}(Y \setminus U)$ is open. But $X \setminus f^{-1}(Y \setminus U) = f^{-1}(U)$. □

Even in this very general setting we can check that some maps are continuous.

Proposition 12.33 *Let X and Y be topological spaces.*

(i) The identity map $\mathrm{id}_X : X \to X$ *is continuous.*
(ii) Let $y_0 \in Y$. The constant map $f : X \to Y$ given by $f(x) = y_0$ is continuous.

Proof. (i) Since id_X is the identity map, $\mathrm{id}_X^{-1}(U) = U$ for every set $U \subset X$. Thus, if U is open, its preimage under id_X is open.

(ii) Let $V \subset Y$ be an open set. If $y_0 \in V$ then $f^{-1}(V) = X$ which is open. If $y_0 \notin V$, then $f^{-1}(V) = \emptyset$ which is also open. □

Proposition 12.34 *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $g \circ f : X \to Z$ is continuous.*

Proof. Let W be an open set in Z. To show that $g \circ f$ is continuous we need to show that $(g \circ f)^{-1}(W)$ is an open set. However, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since g is continuous, $g^{-1}(W)$ is open and since f is continuous $f^{-1}(g^{-1}(W))$ is open. □

As before, we have introduced the notion of continuous function on a level of generality greater than we need. The following result indicates that this abstract definition matches that learned in calculus.

Theorem 12.35 *Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $f : X \to Y$. Then, f is continuous if and only if for every $x \in X$ and any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mathrm{dist}_X(x, y) < \delta$ then $\mathrm{dist}_Y(f(x), f(y)) < \epsilon$.*

In case of a function $f : X \to \mathbf{R}$, the continuity can be characterized by means of lower and upper semicontinuity.

Definition 12.36 A function $f : X \to \mathbf{R}$ is *upper semicontinuous* if the set $\{x \in X \mid f(x) < \alpha\}$ is open for any real α. It is *lower semicontinuous* if the set $\{x \in X \mid f(x) > \alpha\}$ is open for any real α.

We should distinguish between the concept of lower and upper semicontinuous real functions and lower and upper semicontinuous multivalued maps studied in Chapter 6.

Proposition 12.37 *A function $f : X \to \mathbf{R}$ is continuous if and only if it is both upper and lower semicontinuous.*

Proof. Since $\{x \in X \mid f(x) < \alpha\} = f^{-1}((-\infty, \alpha))$, it is clear that any continuous function is upper semicontinuous. The same argument shows that it also is lower semicontinuous. Let f be both lower and upper semicontinuous. Then the inverse image of any open interval (α, β) by f is

$$\{x \in X \mid f(x) > \alpha\} \cap \{x \in X \mid f(x) < \beta\}$$

so it is open. From that it follows that the inverse image of any open set is open. □

We now return to the question of comparing topological spaces. To say that two topological spaces are equivalent is seems natural to require that both objects, the sets and the topologies, be equivalent. On the level of set theory the equivalence of sets is usually taken to be the existence of a bijection. To be more precise, let X and Y be sets. A function $f : X \to Y$ is an *injection* if for any two points $x, z \in X$, $f(x) = f(z)$ implies that $x = z$. f is a *surjection* if for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$. If f is both an injection and a surjection then it is a *bijection*. If f is a bijection then one can define an inverse map $f^{-1} : Y \to X$.

Definition 12.38 Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. A bijection $f : X \to Y$ is a *homeomorphism* if and only if both f and f^{-1} are continuous. We say that X is *homeomorphic* to Y and we write $X \cong Y$ if and only if there exists a homeomorphism $f : X \to Y$.

Proposition 12.39 *Homeomorphism defines an equivalence relation on topological spaces.*

Proof. We need to show that the relation $X \cong Y$ is reflexive, symmetric and transitive.

To see that it is reflexive, observe that given any topological space X the identity map $\mathrm{id}_X : X \to X$ is a homeomorphism from X to X.

Assume that X is homeomorphic to Y. By definition this implies that there exists a homeomorphism $f : X \to Y$. Observe that $f^{-1} : Y \to X$ is also a homeomorphism and hence Y is homeomorphic to X. Thus, homeomorphism is a symmetric relation.

Finally, Proposition 12.34 shows that homeomorphism is a transitive relation, that is if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z. □

Thus, using Theorem 12.35 we can easily show that a variety of simple topological spaces are homeomorphic.

Proposition 12.40 *The following topological spaces are homeomorphic:*

(i) $\mathbf{R}$,
(ii) (a, ∞) *for any* $a \in \mathbf{R}$,
(iii) $(-\infty, a)$ *for any* $a \in \mathbf{R}$,
(iv) (a, b) *for any* $-\infty < a < b < \infty$.

Proof. We begin by proving that $\mathbf{R}$ and (a, ∞) are homeomorphic. Let $f : \mathbf{R} \to (a, \infty)$ be defined by

$$f(x) = a + e^x.$$

This is clearly continuous. Furthermore, $f^{-1}(x) = \ln(x-a)$ is also continuous.

Observe that $f : (a, \infty) \to (-\infty, -a)$ given by $f(x) = -x$ is a homeomorphism. Thus, any interval of the form $(-\infty, b)$ is homeomorphic to $(-b, \infty)$ and hence to $\mathbf{R}$.

Finally, to see that (a, b) is homeomorphic to $\mathbf{R}$ observe that $f : (a, b) \to \mathbf{R}$ given by

$$f(x) = \ln\left(\frac{x-a}{b-x}\right)$$

is continuous and has a continuous inverse given by $f^{-1}(x) = (be^y + a)/(1+e^y)$. □

Inverses of the most elementary functions are rarely given by explicit elementary formulas so it would be nice to know weather or not an inverse of a given continuous bijective map is continuous without knowing it explicitly. In a vector calculus course one we learn the Inverse Function Theorem which can often be applied for that purpose since any differentiable map is continuous. That theorem, however, states about maps on open subsets of $\mathbf{R}^d$ and about the local invertibility but we often want to compare closed subsets. Here is a topological "partner" of the Inverse Function Theorem.

Theorem 12.41 *Let X, Y be closed and bounded subsets of $\mathbf{R}^d$. If $f : X \to Y$ is continuous and bijective then its inverse $f^{-1} : Y \to X$ also is continuous.*

Proof. The proof of this theorem requires the notion of *compactness* and we postpone it to the last section, Exercise 12.32. □

Proving that two spaces are homeomorphic is not always so elementary as in the previous examples.

Example 12.42 In Example 12.30 we showed that the closed unit balls $\bar{B}_0^2$ and $\bar{B}_2^2$ in $\mathbf{R}^2$ are homeomorphic. Without going into details we shall indicate how to construct a homeomorphism in arbitrary dimension.

The boundary of $\bar{B}_0 = B_0^d$ is the sphere

$$S_0^{d-1} = \{x \in [-1, 1]^d \mid x_i = -1 \text{ or } x_i = 1 \text{ for some } i\}$$

and the boundary of $\bar{B}_2 = \bar{B}_2^d$ is the sphere

$$S_2^{d-1} = \{x \in \mathbf{R}^d \mid \sum_{i=1}^{d} x_i^2 = 1\}.$$

Any $x \neq 0$ in $\bar{B}_0$ can be uniquely written as $x = t(x)y(x)$ where $y(x)$ is in the intersection of S_0^{d-1} with the half-line emanating from the origin and passing through x and where $t(x) \in (0,1]$. It can be proved that the map $f : \bar{B}_0 \to \bar{B}_2$ given by

$$f(x) := \begin{cases} t(x)\frac{x}{\|x\|} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a homeomorphism. Moreover, the restriction of this map to S_0^{d-1} is a homeomorphism of that set onto the sphere S_2^{d-1}.

Exercises

12.15 Referring to Example 12.30:

(a) Write down the inverse function for f.
(b) Prove that f is a continuous function.

12.16 Prove Theorem 12.35.

12.17 Construct a homeomorphism from any interval $[a, b]$ onto $[-1, 1]$.

12.18 Show that the boundary Γ^{d-1} of the unit square defined in Exercise 12.13 is homeomorphic to the unit sphere S_0^{d-1} (with respect to the supremum norm) defined in Example 12.42.

12.19 Construct a homeomorphism from the triangle

$$T := \{(s,t) \in \mathbf{R}^2 \mid s + t \leq 1, s \geq 0, t \geq 0\}$$

onto the square $[0, 1]^2$.

12.20 We say that a topological space X has the *fixed point property* if every continuous self-map $f : X \to X$ has a fixed point, i.e. a point $x \in X$ such that $f(x) = x$.
a) Show that the fixed point property is a topological property, i.e. that it is invariant under a homeomorphism.
b) Show that any closed bounded interval $[a, b]$ has the fixed point property.

12.21 * Complete the proof given in Example 12.42 in the case $d = 3$.

12.4 Connectedness

One of the most fundamental global properties of a topological space is whether or not it can be broken into two distinct open subsets. The following definition makes this precise.

Definition 12.43 Let X be a topological space. X is *connected* if the only subsets of X that are both open and closed are $\emptyset$ and X. If X is not connected then it is *disconnected*.

Example 12.44 Let $X = [-1,0) \cup (0,1] \subset \mathbf{R}$. Then X is a disconnected space since by Example 12.26 $[-1,0)$ and $(0,1]$ are both open and closed in the subspace topology.

While it is easy to produce examples of disconnected spaces proving that a space is connected is more difficult. Even the following intuitively obvious result is fairly difficult to prove.

Theorem 12.45 *Any interval in* $\mathbf{R}$ *is connected.*

Hints as to how to prove this theorem can be found in Exercise 12.22 or the reader can consult [37]).

A very useful theorem is the following.

Theorem 12.46 *Let* $f : X \to Y$ *be a continuous function. If* X *is connected, then so is* $f(X) \subset Y$.

Proof. Let $Z = f(X)$. Suppose that Z is disconnected. Then there exists a set $A \subset Z$, where $A \neq \emptyset$ and $A \neq Z$, that is both open and closed. Since f is continuous, $f^{-1}(A)$ is both open and closed. But $f^{-1}(A) \neq \emptyset, X$ which contradicts the assumption that X is connected. □

We can now prove one of the more fundamental theorems from topology that you made use of in your calculus class.

Theorem 12.47 [Intermediate Value Theorem (Darboux)] *If* $f : [a,b] \to \mathbf{R}$ *is a continuous function and if* $f(a) > 0$ *and* $f(b) < 0$, *then there exists* $c \in [a,b]$ *such that* $f(c) = 0$.

Proof. The proof is by contradiction. Assume that there is no $c \in [a,b]$ such that $f(c) = 0$. Then

$$f([a,b]) \subset (-\infty,0) \cup (0,\infty).$$

Let $U = (-\infty,0) \cap f([a,b])$ and $V = (0,\infty) \cap f([a,b])$. Using the subspace topology, U and V are open sets and $f([a,b]) = U \cup V$. Since $f(a) > 0$ and $f(b) < 0$, U and V are not trivial. Therefore, $f([a,b])$ is disconnected contradicting Theorems 12.45 and 12.46. □

The definition of connectedness appears difficult to most of beginners in topology since the notion of a subset which is both open and closed violates their intuitions. We below present a related concept which is closer to intuitions.

Definition 12.48 A topological space X is *path connected* if for every two points $x, y \in X$ there exists a continuous map on an open interval $f : [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$. Such a map is called a *path joining* x *to* y.

Any interval obviously is path connected. A special case of a path connected set is convex set:

Definition 12.49 A subset C of $\mathbf{R}^d$ is called *convex* if, given any two points $x, y \in C$, the line segment

$$[x, y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\}$$

joining x to y is contained in C.

Example 12.50 The unit circle S^1_2 in $\mathbf{R}^2$ is not convex but it is path connected. Indeed, let $x, y \in S^1_2$. In polar coordinates, $x = (\cos\alpha, \sin\alpha)$ and $y = (\cos\beta, \sin\beta)$. Without a loss of generality, $\alpha \leq \beta$. Then $\sigma : [\alpha, \beta] \to S^1_2$ given by $\sigma(\theta) = (\cos\theta, \sin\theta)$ is a path joining x to y.

Theorem 12.51 *Any path connected space is connected.*

Proof. We argue by contradiction. Suppose that X is disconnected and let $B \subset X$ be non-empty, different from X, both open and closed. Then there exist two points $x \in B$ and $y \in X \setminus B$. Let $\sigma : [a, b] \to X$ be a path joining x to y and let $A := \sigma^{-1}(B)$. Since σ is continuous, A is both open and closed in $[a, b]$. Since $\sigma(a) = x$, $a \in A$ so $A \neq \emptyset$. Without loss of generality we may assume that b is the only point of $\sigma^{-1}(y)$, otherwise we could choose the smallest $c \in [a, b]$ such that $\sigma(c) = y$ and restrict σ to the interval $[a, c]$. Hence $\{b\} = \sigma^{-1}(y) \subset X \setminus A$ so $A \neq [a, b]$. That contradicts Theorem 12.45. □

A connected space does not need to be path connected but such spaces are quite "pathological" (see Exercise 12.28). For a large class of spaces such as geometric polyhedra, the two definitions are equivalent: we shall return to this topic when we discuss graphs and cubical sets.

Proving that two spaces are not homeomorphic is often harder since showing that there exists no homeomorphism from one onto another seems non constructive at the first sight. Knowing *topological properties*, i.e. properties of spaces which are preserved by homeomorphisms comes with help here. By Theorem 12.46, connectedness is a topological property. This gives a simple criterion for distinguishing between two spaces.

Example 12.52 The half-closed interval $(0, 1]$ is not homeomorphic to the open interval $(0, 1)$. We will argue by contradiction. Suppose that $f : (0, 1] \to (0, 1)$ is a homeomorphism and let $t := f(1)$. Then the restriction of f to $(0, 1)$ is a homeomorphism of $(0, 1)$ onto the set $(0, t) \cup (t, 1)$. That is impossible since the first set is connected and the second is not.

Definition 12.53 A subset A of a topological space X is called *connected* if A is connected as a topological space with subset topology.

Theorem 12.54 *Assume $\{X_\iota\}_{\iota \in J}$ is a family of connected subsets of a topological space Z. If $\bigcap_{\iota \in J} X_\iota \neq \emptyset$ then $\bigcup_{\iota \in J} X_\iota$ is connected.*

Proof. Let $X := \bigcup_{\iota \in J} X_\iota$ and let F be a non-empty subset of X which is both open and closed. Let $x \in F$. Then $x \in X_{\iota_0}$ for some $\iota_0 \in J$. It follows that $F \cap X_{\iota_0}$ is non-empty and both open and closed in X_{ι_0}. Since X_{ι_0} is connected, it must be $F \cap X_{\iota_0} = X_{\iota_0}$, i.e. $X_{\iota_0} \subset F$. Recalling that $\bigcap_{\iota \in J} X_\iota$ is non-empty, we get $F \cap X_\iota \neq \emptyset$ for every $\iota \in J$. This means, by the same argument as in the case of X_{ι_0}, that $X_\iota \subset F$ for every $\iota \in J$. Hence $X = F$, i.e. X is connected. □

Theorem 12.55 *Every rectangle in $\mathbf{R}^d$, i.e. a product of closed bounded intervals is connected.*

Proof. Let $X := [a_1, b_1] \times [a_2, b_2] \times \cdots [a_d, b_d]$. Given any points $x, y \in X$, the function $f : [0,1] \to \mathbf{R}^d$ given by $f(t) = (1-t)x + ty$ is a continuous path with values in X connecting x to y. Hence X is path connected. □

It is easy to see that if the number of connected components is finite, it is a topological property, i.e. if X has k connected components than the same is true about any space homeomorphic to X.

Exercises

12.22 Let A and B be two disjoint nonempty open sets in $I = [0,1]$. The following arguments will show that $I \neq A \cup B$ and therefore that I is a connected set.

Let $a \in A$ and $b \in B$, then either $a < b$ or $a > b$. Assume without loss of generality that $a < b$.

(a) Show that the interval $[a,b] \subset I$.

Let $A_0 := A \cap [a,b]$ and $B_0 := B \cap [a,b]$.

(b) Show that A_0 and B_0 are open in $[a,b]$ under the subspace topology.

Let c be the least upper bound for A_0, i.e.

$$c := \inf\{x \in \mathbf{R} \mid x > y \text{ for all } y \in A_0\}.$$

(c) Show that $c \in [a,b]$.
(d) Show $c \notin B_0$. Use the fact that c is the least upper bound for A_0 and that B_0 is open.
(e) Show that $c \notin A_0$. Again use the fact that c is the least upper bound for A_0 and that A_0 is open.

Finally, observe that $c \in I$, but $c \notin A_0 \cup B_0$ and therefore, that $I \neq A_0 \cup B_0$.

12.23 Prove the converse of Theorem 12.45: If $X \subset \mathbf{R}$ is connected then X is an interval (by interval we mean any type of of bounded interval, half-line, or $\mathbf{R}$) .

12.24 Use Theorem 12.45 to show that S^1_2 is connected.

12.25 Let A and B be connected sets. Assume that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected.

12.26 Show that the unit circle $S_2^1 = \{x \in \mathbf{R}^2 \mid \|x\|_2 = 1\}$ is not homeomorphic to an interval (whether it is closed, open or neither).
Hint: Use arguments similar to that in Example 12.52.

12.27 * A *simple closed curve* in $\mathbf{R}^d$ is an image of an interval $[a, b]$ under a continuous map $\sigma : [a, b] \to \mathbf{R}^d$ (called *path*) such that $\sigma(s) = \sigma(t)$ for any $s < t, s, t \in [a, b]$ if and only if $s = a$ and $t = b$. Prove that any simple closed curve is homeomorphic to the unit circle S_2^1.
Hint: Recall the path in Example 12.50 and use Theorem 12.41.

12.28 * Let $X \subset \mathbf{R}^d$ be the union of the interval $\{0\} \times [-1, 1]$ and the set Σ given by

$$\Sigma := \{(s, t) \in \mathbf{R}^2 \mid t = \sin \frac{1}{s}, 0 < s \leq \frac{1}{2\pi}\}.$$

Draw X. Show that X is connected but not path connected.

12.5 Limits and Compactness

Limits of sequences play a central role in any textbook on analysis, numerical analysis, and metric space topology. So far we purposely avoided speaking about limits so to emphasize our combinatorial approach to topology. This notion, however, becomes hard to avoid when discussing compactness and properties of continuous functions on compact sets. Due to space limitations we can only provide a very brief presentation of the topic. We assume that the reader is familiar with the calculus of limits of sequences of real numbers and we refer to standard textbooks in topology such as [37] for two omitted proofs.

Definition 12.56 Let (X, dist) be a metric space. A sequence

$$(x_n)_{n \in N} = (x_1, x_2, x_3, \ldots)$$

(or shortly (x_n)) of points of X *converges* to $x \in X$ if and only if for every $\epsilon > 0$ there exists n_0 such that, for any $n > n_0$, $\text{dist}(x_n, x) < \epsilon$. The point x is called the *limit* of (x_n) and we write either $(x_n) \to x$ or $\lim_{n\to\infty} x_n = x$.

Example 12.57 Let $X = \mathbf{R}$

1. $\lim_{n\to\infty}(\frac{1}{n}) = 0$;
2. The sequence $(n)_{n \in N} = (1, 2, 3, \ldots)$ diverges;
3. The sequence $(x_n) = ((-1)^n) = (-1, 1, -1, 1, \ldots\ldots\ldots)$ diverges. However one can extract from it two convergent subsequences $(x_{2n}) = (1) \to 1$ and $(x_{2n-1}) = (-1) \to -1$.

Proposition 12.58 *Let $(x(n))_{n \in N}$ be a sequence of points*

$$x(n) = (x_1(n), x_2(n), \ldots x_d(n))$$

in $\mathbf{R}^d$. The sequence $(x(n)_{n \in N})$ converges in $\mathbf{R}^d$ if and only if $(x_i(n))_{n \in N}$ converges in $\mathbf{R}$ for every $i = 1, 2, \ldots d$.

Proof. We have chosen dist_0 as the standard metric in $\mathbf{R}^d$. Thus the the condition $\operatorname{dist}_0(x(n), x) < \epsilon$ is equivalent to $|x_i(n) - x_i| < \epsilon$ for all $i = 1, 2, \ldots d$. Suppose that $(x(n)) \to x$, take any $\epsilon > 0$ and let n_0 be as in the definition of convergence. Then $|x_i(n) - x_i| < \epsilon$ for all $n > n_0$ and all i so $x_i(n) \to x_i$ for all i. Conversely, if, given $\epsilon > 0$, n_i is such that $\operatorname{dist}(x_i(n), x) < \epsilon$ for any $n > n_i$, it is enough to take $n_0 := \min\{n_1, n_2, \ldots n_d\}$ and the condition for the convergence of $(x(n))$ follows. □

Many topological notions discussed in the previous sections can be characterized in terms of convergent sequences. Here are some most useful facts:

Proposition 12.59 *Let (X, dist) be a metric space and $A \subset X$ then A is closed if and only if for every sequence (a_n) of points in A, $(a_n) \to x \in X$ implies $x \in A$.*

Proof. We argue by contradiction. Suppose that A is closed and let $(a_n) \to x$ where $a_n \in A$ but $x \notin A$. Since $X \setminus A$ is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap A = \emptyset$. That contradicts $\operatorname{dist}(a_n, x) < \epsilon$ for all sufficiently big n.

Conversely, suppose all convergent sequences in A have limits in A. We show that $U := X \setminus A$ is open. If not, then there exists $x \in U$ such that every ball $B(x, r)$ intersects A. For any $n \in \mathbf{N}$ choose $a_n \in B(x, \frac{1}{n}) \cap A$. Then $(a_n) \to x \notin A$, a contradiction. □

Proposition 12.60 *Let $(X, \operatorname{dist}_X)$ and $(Y, \operatorname{dist}_Y)$ be metric spaces and $f : X \to Y$. Then f is continuous if and only if for every $x \in X$ and every sequence $(x_n) \to x$ we have $(f(x_n)) \to f(x)$.*

Proof. Suppose that f is continuous, let $(x_n) \to x$ and take any $\epsilon > 0$. Let δ be as in Theorem 12.35 and let n_0 be such that $\operatorname{dist}_X(x_n, x) < \delta$ for all $n > n_0$. Then $\operatorname{dist}_Y(f(x_n), f(x)) < \epsilon$ for all those n so $(f(x_n)) \to f(x)$.

Conversely, suppose the second condition is satisfied but f is not continuous. Let x and ϵ be such that the conclusion of Theorem 12.35 fails. Then for any $n \in \mathbf{N}$ there exists x_n such that $\operatorname{dist}_X(x_n, x) < \frac{1}{n}$ but $\operatorname{dist}_Y(f(x_n), f(x)) > \epsilon$. Then $(x_n) \to x$ but $f(x_n) \not\to f(x)$. □

The following theorem displays the fundamental property of the set $\mathbf{R}$ of real numbers which distinguishes it, for example, from the set $\mathbf{Q}$ of rational numbers. We refer the reader to any textbook in real analysis for its proof.

Theorem 12.61 [Bolzano-Weierstrass Theorem] *Every bounded sequence of real numbers contains a subsequence convergent to a real number.*

We shall now give the definition of a compact metric space.

Definition 12.62 A metric space X is called *compact* if every sequence in X contains a convergent subsequence.

The above definition is not the most general one but the most elementary one. The general definition for topological spaces uses the concept of covering which will be presented at the end. Here is a generalization of Bolzano-Weierstrass theorem for sequences in $\mathbf{R}^d$.

Theorem 12.63 *A subset X of $\mathbf{R}^d$ is compact if and only if it is closed and bounded. In particular, every cubical set is compact.*

Proof. Supposed that X is closed and bounded in $\mathbf{R}^d$ and let $(x(n))_{n\in N}$ be a sequence in X. Then $(x_i(n))_{n\in N}$ is a bounded sequence in $\mathbf{R}$ for any $i = 1, 2, \ldots d$. By Bolzano-Weierstrass Theorem, $(x_1(n))$ contains a convergent subsequence $(x_1(n_k)) \to x_1$. We apply the same argument to $x_2(n_k)$ and get $(x_2(n_{k_l})) \to x_2$. Since a subsequence of a convergent sequence converges to the same limit, we also have $(x_1(n_{k_l})) \to x_1$. We continue this way up to $i = d$. By suppressing multiple indices, we may assume that $(x_i(n_m)) \to x_i$. By Proposition 12.58, $(x_{n_m}) \to x$. Since X is closed, by Proposition 12.59, $x \in X$. We proved that X is compact.

We leave the proof of the reverse implication as an exercise. □

We present below some some useful properties of compact sets.

Definition 12.64 Let $(X, \operatorname{dist}_X)$ and $(Y, \operatorname{dist}_Y)$ be metric spaces. A function $f : X \to Y$ is called *uniformly continuous* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for any $x, y \in X$, if $\operatorname{dist}_X(x, y) < \delta$, then $\operatorname{dist}_Y(f(x), f(y)) < \epsilon$. In other words, f is uniformly continuous if δ in Theorem 12.35 does not depend on the choice of x.

Theorem 12.65 *Let $(X, \operatorname{dist}_X)$ and $(Y, \operatorname{dist}_Y)$ be metric spaces. If X is compact, than every continuous function $f : X \to Y$ is uniformly continuous.*

Proof. We argue by contradiction. Suppose that there exists $\epsilon > 0$ such that for any $\delta > 0$ there is a pair of points $x, y \in X$ such that $\operatorname{dist}_X(x, y) < \delta$ but $\operatorname{dist}_Y(f(x), f(y) \geq \epsilon$. By taking $\delta = \frac{1}{n}$ we get a pair of sequences (x_n) and (y_n) such that $\operatorname{dist}_X(x_n, y_n) < \frac{1}{n}$ but $\operatorname{dist}_Y(f(x_n), f(y_n)) \geq \epsilon$. By compactness, we obtain subsequences $(x_{n_k}) \to x$ and $(y_{n_k}) \to y$. But

$$\operatorname{dist}_X(x, y) \leq \operatorname{dist}_X(x, x_{n_k}) + \operatorname{dist}_X(x_{n_k}, y_{n_k}) + \operatorname{dist}_X(y_{n_k}, y) \to 0,$$

hence $\operatorname{dist}_X(x, y) = 0$ so $x = y$. By Proposition 12.60,

$$\epsilon \leq \operatorname{dist}_Y(f(x_{n_k}), f(y_{n_k})) \leq \operatorname{dist}_Y(f(x_{n_k}), f(x)) + \operatorname{dist}_Y(f(y_{n_k}), f(x)) \to 0$$

so $\epsilon = 0$ which contradicts that $\epsilon > 0$. □

Proposition 12.66 *Let $f : X \to Y$ be continuous function. If X is compact then $f(X)$ is also compact.*

Proof. Let (y_n) be a sequence in $f(X)$ and let $x_n \in f^{-1}(y_n)$. Then (x_n) contains a convergent subsequence $(x_{n_k}) \to x$. By Proposition 12.60, $(y_{n_k}) = (f(x_{n_k})) \to f(x)$. □

The proof of the following is left as an exercise.

Theorem 12.67 *Let X be a compact metric space. Any continuous function $f : X \to \mathbf{R}$ assumes its minimum and maximum in X, i.e. there are points $x_0, x_1 \in X$ such that $f(x_0) \leq f(x)$ for all $x \in X$ and $f(x_1) \geq f(x)$ for all $x \in X$.*

Definition 12.68 Let X be a topological space. A family $\{U_\iota\}_{\iota \in J}$ of subsets of X is called *covering* of X if it's union is all X. It is called *open covering* if all U_ι are open. A *subcovering* of $\{U_\iota\}_{\iota \in J}$ is a subfamily of it which is a covering of X.

The statement of the next theorem is actually used as a definition of a compact space in courses of general topology. The proof that the two conditions of compactness are equivalent in the case of metric spaces is too involved for the scope of this presentation and we refer the reader to any textbook in general topology.

Theorem 12.69 A metric space X is compact if and only if every open covering of X contains a finite open subcovering.

Exercises

12.29 Let X be a metric space and consider $f : X \to \mathbf{R}^d$,

$$f(x) = (f_1(x), f_2(x), \ldots f_d(x)).$$

Show that f is continuous if and only if the coordinate functions $f_i(x)$ are continuous for all i.

12.30 Prove that any compact subset of $\mathbf{R}^d$ is closed and bounded.

12.31 Prove Theorem 12.67.

12.32 Let X, Y be metric spaces, X compact and $f : X \to Y$ a continuous bijection. Prove that f^{-1} is continuous.

12.33 Let $S_1 \supset S_2 \supset S_3 \supset \cdots$ be a decreasing sequence of nonempty compact sets. Prove that $S := \bigcap_{i=1}^{\infty} S_i$ is a nonempty compact set.

13

Algebra

13.1 Abelian Groups

13.1.1 Algebraic Operations

Algebra is devoted to the study of the properties of operations. The operations we use in every day life are addition, subtraction, multiplication and division of numbers. Formally, a *(binary) operation* on a set G is a mapping $q : G \times G \to G$. Rather than writing the operation in this functional form, e.g. $q(a, b)$, one typically selects a symbol, for instance a diamond, and uses a notation such as $a \diamond b$. Frequently the symbol is borrowed from the operations on numbers, especially when this emphasizes some similarities. Since we want to study similarities to addition of numbers, we will use the symbol $+$ in the sequel.

The fundamental property of the addition of numbers is that for any numbers a, b, c

$$a + (b + c) = (a + b) + c, \tag{13.1}$$

i.e. if we have to add three numbers than it does not matter in which order we perform the addition. This is more or less obvious for addition (and also multiplication) of numbers but needn't be true for other operations. If an operation on a set G satisfies (13.1) for any $a, b, c \in G$, then we say that it is *associative*. An example of an operation which is not associative is subtraction of numbers.

Another property, which is obvious for addition of numbers, is that it is commutative. We say that an operation on a set G is *commutative* if for any $a, b \in G$

$$a + b = b + a. \tag{13.2}$$

We say that $e \in G$ is the *identity element* if for all $a \in G$

$$a + e = e + a = a. \tag{13.3}$$

It is easy to see that if such an element exists, then it is unique. In the case of addition of numbers the identity element is zero. For this reason 0 is traditionally used to denote the identity element and we do so in the sequel.

We say that $a' \in G$ is an *inverse* of $a \in G$ if

$$a + a' = a' + a = 0. \tag{13.4}$$

This of course makes sense only if the identity element exists. If the operation is associative, then the inverse element is unique, because if a'' is another inverse of a then

$$a'' = 0 + a'' = (a' + a) + a'' = a' + (a + a'') = a' + 0 = a'.$$

In the case of addition of numbers it is obvious that the inverse of a number a is just $-a$. For this reason $-a$ is traditionally used to denote the inverse element of a and we do so in the sequel.

So far we spoke of addition of numbers without stating precisely what set of numbers we consider. Not every set is acceptable if we want to consider addition as an operation. For instance the interval $[0, 1]$ is not good, because the fractions $0.6, 0.7 \in [0, 1]$ but $1.3 = 0.6 + 0.7 \notin [0, 1]$. However, there are several sets of numbers, where addition is an operation, for instance the set of all real numbers $\mathbf{R}$, the set of rational numbers $\mathbf{Q}$, the set of integers $\mathbf{Z}$, the set of natural numbers $\mathbf{N}$ and the set of non-negative integers $\mathbf{Z}^+$. Notice, however, that in the case of the addition in $\mathbf{N}$ there is no identity element, because $0 \notin \mathbf{N}$. Also, although $0 \in \mathbf{Z}+$, the addition in $\mathbf{Z}^+$ does not admit the existence of inverse elements. For instance $2 \in \mathbf{Z}^+$ but $-2 \notin \mathbf{Z}^+$.

13.1.2 Groups

As we will see in the sequel, there are several operations on sets other than numbers, which have the same properties as the four discussed properties of the addition of numbers: associativity, commutativity, the existence of identity element and the existence of inverse elements. Whatever we are able to prove for the addition of numbers utilizing only these four properties will be also true for the other operations as long as they have the same four properties. Therefore it seems to be useful to do some abstraction and give a name to a set with an operation satisfying such four properties.

For this reason we give the following definition

Definition 13.1 An *abelian group* is a pair $(G, +)$, consisting of a set G and a binary operation $+$ defined on G and satisfying the following four axioms:

1. the operation is associative
2. the operation is commutative
3. the operation admits the identity element
4. the operation admits the inverse for every element of G

In practice the operation is often clear from the context. In such a case we often simply say that G is a group. Also, when not explicitly stated otherwise, we denote the operation by $+$.

The word "abelian" refers to the commutativity property. If G satisfies all other axioms but not necessarily commutativity, it is simply called a *group*. Since throughout this text we shall only consider abelian groups, we shall often say "group" meaning "abelian group".

Example 13.2 The sets $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ with the standard addition operation are all abelian groups. The sets $\mathbf{Z}^+$ and $\mathbf{N}$ are not groups.

The preceding example may tempt us to say that a group must contain an infinite number of elements. That this needn't be true shows the following example.

Example 13.3 Let $G = \{0, 1, 2\}$ and let the operation $+$ in G be defined by the following table

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

It is straightforward to see that the operation is associative, commutative, 0 is the identity element, the inverse of 0 is 0, the inverse of 1 is 2 and the inverse of 2 is 1. Therefore we have an example of an abelian group.

Actually, the last example is a special case of the following more general example.

Example 13.4 Given a positive integer n put $\mathbf{Z}_n := \{0, 1, 2, \ldots n-1\}$ and define an operation $+_n : \mathbf{Z}_n \times \mathbf{Z}_n \to \mathbf{Z}_n$ by

$$a +_n b := (a + b) \bmod n,$$

where $(a+b) \bmod n$ is the remainder of $a + b \in \mathbf{Z}$ after division by n, i.e. the smallest integer $c \geq 0$ such that $a + b - c$ is divisible by n. One can check that $(Z_n, +_n)$ is a group. Its identity element is just 0. The inverse of $a \in \mathbf{Z}_n$ is $n - a$. Notice that the previous example is just $(Z_3, +_3)$. In the sequel we shall abandon the subscript n in $+_n$ when it is clear from the context that we mean the addition in $\mathbf{Z}_n$ and not in $\mathbf{Z}$.

The smallest possible group consists of just one element, which must be the identity element of this group, i.e. 0. This group is called the *trivial* group and is denoted by $\{0\}$ or simply 0.

Given abelian groups $G_1, G_2, \ldots, G_n$, their product

$$\prod_{i=1}^{n} G_i = G_1 \times G_2 \times \cdots \times G_n \tag{13.5}$$

is an abelian group with the operation being coordinate-wise addition

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

One can easily check that the identity element in this group is $(0, 0, \ldots, 0)$ and the inverse of $(a_1, a_2, \ldots a_n)$ is $(-a_1, -a_2, \ldots - a_n)$.

Example 13.5 $\mathbf{R}^n$ and $\mathbf{Z}^n$ with the addition operation defined above are examples of abelian groups.

The elements of $\mathbf{R}^n$ are called *vectors* or *number vectors* to distinguish them from the abstract vectors we will introduce later. For $\mathbf{x} = (x_1, x_2, \ldots x_n) \in \mathbf{R}^n$ the number x_i will be called the $i - th$ coordinate of $\mathbf{x}$. Number vectors will be used frequently in this book. Throughout the book we adopt the convention that whenever we use a lower case bold letter, it denotes a number vector whose coordinates are denoted by the same but boldless letters.

$$\mathbf{x} = (x_1, x_2, \ldots x_n).$$

If all the numbers x_i are real numbers, than we call $\mathbf{x}$ a *real vector* or just a vector. If all x_i are integers, we call it an *integer vector*. Obviously integer vectors are elements of $\mathbf{Z}^n$.

Definition 13.6 Let G be an abelian group with the binary operation $+$. A nonempty subset $H \subset G$ is a *subgroup* of G if:

1. $0 \in H$,
2. for every $a \in H$ its inverse $-a \in H$,
3. H is *closed* under $+$, i.e. given $a, b \in H$, $a + b \in H$.

Example 13.7 $(\mathbf{Z}, +)$ is a subgroup of $(\mathbf{Q}, +)$ which in turn is a subgroup of $(\mathbf{R}, +)$. $(\mathbf{Z}^d, +)$ is a subgroup of $(\mathbf{R}^d, +)$.

Proposition 13.8 *Let H be a subset of G with the property that if $a, b \in H$ then $a - b \in H$. Then H is a subgroup of G.*

13.1.3 Cyclic Groups and Torsion Subgroup

Let G be a group and let $g \in G$. Given $m \in \mathbf{Z}$, we use the notation

$$mg := \underbrace{g + g + \cdots + g}_{m\ terms} \tag{13.6}$$

to denote the sum of g with itself m times. If m is a negative integer, then this should be interpreted as the m-fold sum of $-g$.

For $a \in G$ put

$$\langle a \rangle := \{na \in G \mid n \in \mathbf{Z}\}.$$

It is straightforward to verify that $\langle a \rangle$ is a subgroup of G. The group G is called *cyclic* if there exists $a \in G$ such that $G = \langle a \rangle$. In particular, if $a \in G$ then $\langle a \rangle$ is a cyclic subgroup of G.

The *order* of G, denoted by $|G|$, is the number of elements of G. Thus $|\mathbf{Z}| = \infty$ and $|\mathbf{Z}_n| = n$. The *order of an element* $a \in G$, denoted by $o(a)$, is the smallest positive integer n such that $na = 0$, if it exists, and ∞ if not. Observe that $|\langle a \rangle| = o(a)$.

Proposition 13.9 *The set of all elements in G of finite order is a subgroup of G.*

The proof of this proposition is left as an exercise.

Definition 13.10 The subgroup of G of all elements of finite order is called the *torsion subgroup* of G. This subgroup is denoted by $T(G)$.

Example 13.11 The addition table for $\mathbf{Z}_6$ is as follows:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Using the table it is easy to check that: 0 has order 1, 1 and 5 have order 6 thus each of them generates the whole group, 2 has order 3 and 3 has order 2. Note the relation between the divisors of 6 and orders of elements of $\mathbf{Z}_6$.

Example 13.12 In the group $\mathbf{Z}_2^2 = \mathbf{Z}_2 \times \mathbf{Z}_2$ of order 4, all 3 nonzero elements $(0,1)$, $(1,0)$, and $(1,1)$ have order 2. Thus this is not a cyclic group. Consider the group $\mathbf{Z}_2 \times \mathbf{Z}_3$. Here are the orders of its elements:

$$o(0) = 1,\ o((1,0)) = 2,\ o((0,1)) = o((0,2)) = 3,\ o((1,1)) = o((1,2)) = 6\ .$$

Thus $\mathbf{Z}_2 \times \mathbf{Z}_3$ is cyclic of order 6, generated by $(1,1)$ and by $(1,2)$. The notion of isomorphism introduced in the next section will permit us to identify this group with $\mathbf{Z}_6$. The same consideration applies to $\mathbf{Z}_n \times \mathbf{Z}_m$ where n and m are relatively prime (see exercises).

We end this section with the following observation

Lemma 13.13 *Any subgroup of a cyclic group is cyclic.*

Proof. Let G be a cyclic group and let H be a subgroup of G. We may assume that $H \neq 0$. Let $a \in G$ be such that $G = \langle a \rangle$. Then the nonzero element of H has the form ka for some non-zero integer $k \in \mathbf{Z}$. Let k_0 be the smallest positive integer such that $k_0 a \in H$. Then obviously $nk_0 a \in H$ for all integers n. We need to show that all elements of H are of this form. Indeed, if not, then there exists $h \in H$ of the form $h = (nk_0 + r)a$ where $0 < r < k_0$. Since $nk_0 a \in H$, we get $ra \in H$, which contradicts the minimality of k_0. $\square$

13.1.4 Quotient Groups

Let H be a subgroup of G and let $a \in G$. The set

$$a + H := \{a + h \ : \ h \in H\}$$

is called a *coset* of H in G. The element a is called its *representative*. Typically a coset will have many different representatives. For example, let $h_0 \in H$, $a \in G$ and $b = a + h_0$, then a and b are representatives for the same coset. The following proposition makes this precise.

Proposition 13.14 *Let H be a subgroup of G and $a, b \in G$. Then*

(a) The cosets $a + H$ and $b + H$ are either equal or disjoint.
(b) $a + H = b + H$ if and only if $b - a \in H$.

Proof. (a) Suppose that $(a + H) \cap (b + H) \neq \emptyset$. Then there exist $h_1,\ h_2 \in H$ such that $a + h_1 = b + h_2$. Hence, for any $h \in H$, $b + h = a + h_1 - h_2 + h \in a + H$ so $b + H \subset a + H$. The reverse inclusion holds by the symmetric argument.

(b) Let $a + H = b + H$ and let $h_1,\ h_2$ be as in (a). Then $b - a = h_1 - h_2 \in H$. Conversely, if $b - a \in H$ then $b + 0 = a + (b - a) \in (b + H) \cap (a + H)$, thus the conclusion follows from (a). □

Writing cosets in the form of $a + H$ is a bit cumbersome, so we shorten it to $[a] := a + H$. Notice that to use this notation it is essential that we know the subgroup H that is being used to form the cosets.

We can define a binary operation on the set of cosets by setting

$$[a] + [b] := [a + b]. \tag{13.7}$$

Observe that $[0] + [a] = [0 + a] = [a]$ so $[0]$ acts like an identity element. Furthermore, $[a] + [-a] = [a + -a] = [0]$, so there are inverse elements. It is also easy to check that this operation is associative and commutative. The only serious issue is whether this new operation is well defined, in other words does it depend on which representative we use.

Proposition 13.15 *Formula (13.7) does not depend on the choice of coset representative used, and therefore, defines a group structure on $\{a + H\}_{a \in G}$.*

Proof. If $a' + H = a + H$ and $b' + H = b + H$ then, by Proposition 13.14, $a' - a \in H$, $b' - b \in H$ and so $(a' + b') - (a + b) = (a' - a) + (b' - b) \in H$. Hence $a' + b' + H = a + b + H$. □

Definition 13.16 The group of cosets described by Proposition 13.15 is called the *quotient group* of G by H and denoted by G/H.

An alternative way of introducing the quotient group is in terms of an equivalence relation. Define the relation $a \sim b$ if and only if $b - a \in H$. Note that this is an *equivalence relation* in G, i.e.

i) $a \sim a$, for all $a \in G$;
ii) $a \sim b \Leftrightarrow b \sim a$, for all $a, b \in G$;
iii) $a \sim b$ and $b \sim c \Rightarrow a \sim c$, for all $a, b, c \in G$.

The *equivalence class* of $a \in G$ is the set of all $b \in G$ such that $b \sim a$. Thus, by Proposition 13.14 the group of cosets is exactly the group of equivalence classes of $a \in G$.

Proposition 13.17 *Let G be a finite group and H its subgroup. Then each coset $a + H$ has the same number of elements. Consequently,*

$$|G| = |G/H| \cdot |H| \, .$$

Proof. The first conclusion is an obvious consequence of the cancellation law for the group addition: $a + h_1 = a + h_2 \Leftrightarrow h_1 = h_2$. The second conclusion is an immediate consequence of the first one and Proposition 13.14(a). □

Example 13.18 Let $G = \mathbf{Z}$ and $H = k\mathbf{Z}$ for some $k \in \mathbf{Z},\ k \neq 0$, the group $G/H = \mathbf{Z}/k\mathbf{Z}$ has k elements $[0], [1], \ldots, [k-1]$. Since the coset $[a+b]$ is also represented by the remainder of the division of $a + b$ by k, this group may be identified with Z_k discussed in the previous section. What "identification" means, will become clear later, when we talk about isomorphisms.

Example 13.19 Let $G = \mathbf{Z}^2$ and $H = \{(-n, n) \ : \ n \in \mathbf{Z}\}$. We may choose coset representatives of the form $(m, 0)$, $m \in \mathbf{Z}$. Since any element $(m, n) \in \mathbf{Z}^2$ can be written as $(m + n)(1, 0) + n(-1, 1) \in (m + n)(1, 0) + H$, we have $G/H = \{[m(1, 0)]\}_{m \in \mathbf{Z}}$. It is easily seen that $[k(1, 0)] \neq [m(1, 0)]$ whenever $k \neq m$, thus there is a bijection between G/H and $\mathbf{Z}$.

Example 13.20 Consider $\mathbf{Z}$ as a subgroup of $\mathbf{R}$ and the quotient $\mathbf{R}/\mathbf{Z}$. Since any real number is an integer translation of a number in the interval $[0, 1)$, $\mathbf{R}/\mathbf{Z}$ is represented by the points of that interval. Moreover there is a bijection between $\mathbf{R}/\mathbf{Z}$ and $[0, 1)$, since no two numbers in that interval may differ by an integer. For any $\alpha, \beta \in [0, 1)$, the coset $[\alpha + \beta]$ is represented in $[0, 1)$ by the fractional part of $\alpha + \beta$. Since $1 \sim 0$, $\mathbf{R}/\mathbf{Z}$ may be visualized as a circle obtained from the interval $[0, 1]$ by gluing 1 to 0.

A very similar example explaining the concept of polar coordinates is the quotient group $\mathbf{R}/2\pi\mathbf{Z}$. The equivalence relation is now $\alpha \sim \beta \Leftrightarrow \beta - \alpha = 2n\pi, n \in \mathbf{Z}$ and the representatives may be searched, for example, in the interval $[0, 2\pi)$. Thus the elements of $\mathbf{R}/2\pi\mathbf{Z}$ may be identified with the points on the circle $x^2 + y^2 = 1$ in the plane, via the polar coordinate θ in $x = \cos\theta,\ y = \sin\theta$.

13.1.5 Direct Sums

Let A and B be subgroups of G. We define their *sum* by

$$A + B := \{c \in G : c = a + b \text{ for some } a \in A, b \in B\} \, . \tag{13.8}$$

We say that G is a *direct sum* of A and B and write

$$G := A \oplus B$$

if $G = A + B$ and the decomposition $c = a + b$ of any $c \in G$ is unique. We have the following simple criterion for a direct sum.

Proposition 13.21 *Let G be the sum of its subgroups A and B. Then $G = A \oplus B$ if and only if $A \cap B = \{0\}$.*

Proof. Suppose that $A \cap B = \{0\}$ and that $c = a_1 + b_1 = a_2 + b_2$ are two decompositions of $c \in G$, $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $a_1 - a_2 = b_2 - b_1 \in A \cap B = \{0\}$ which implies that $a_1 = a_2$ and $b_1 = b_2$. Hence the decomposition is unique. Conversely, let $A \cap B \neq \{0\}$ and let $c \in A \cap B$, $c \neq 0$. Then c can be decomposed as $c = a + b$ in at least two ways: by setting $a := c,\ b := 0$ or $a := 0,\ b := c$. □

In a similar way one defines the sum and direct sum of any family $G_1, G_2, \ldots, G_n$ of subgroups of a given abelian group G. G is the *direct sum* of $G_1, G_2, \ldots, G_n$ if every $g \in G$ can be uniquely written as $a = \sum_{i=1}^n g_i$, where $g_i \in G$ for all $i = 1, 2, \ldots, n$. We write

$$G = \bigoplus_{i=1}^{n} G_i = G_1 \oplus G_2 \oplus \cdots \oplus G_n \ . \tag{13.9}$$

There is a close relation between direct products and direct sums. Let $G = G_1 \times G_2 \times \cdots \times G_n$. We may identify each G_i with the subgroup

$$G_i' := \{0\} \times \cdots \times \{0\} \times \underbrace{G_i}_{\text{i'th place}} \times \{0\} \times \cdots \times \{0\} \ .$$

Then

$$G = G_1' \oplus G_2' \oplus \cdots \oplus G_n'$$

and, for the simplicity of notation, we may write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \ .$$

This identification of direct products and sums will become more formal when we talk about isomorphisms of groups in the next section. When infinite families of groups are considered, their direct sum may only be identified with a subgroup of the direct product consisting of sequences which have zeros in all but finitely many places.

Exercises

13.1 Let G be an abelian group. Prove that if there exists an identity element 0 of an operation then it is unique.

13.2 Prove Proposition 13.8

13.3 Determine the orders of all elements of $\mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_8$

13.4 Prove Proposition 13.9

13.5 (a) Let m, n be relatively prime. Show that $\mathbf{Z}_m \oplus \mathbf{Z}_n$ is cyclic of order mn.
(b) Let $G = \mathbf{Z}_{12} \oplus \mathbf{Z}_{36}$. Express G as a direct sum of cyclic groups whose orders are powers of primes.

13.6 (a) Prove that a group of prime order has no proper subgroup.
(b) Prove that if G is a cyclic group and p is a prime dividing $|G|$, then G contains an element of order p.

13.2 Fields and Vector Spaces

13.2.1 Fields

Is the multiplication of numbers an operation which can be used to build a group? The multiplication of numbers is associative and commutative. Since for any number a

$$1 \cdot a = a \cdot 1 = a,$$

the identity element for multiplication of numbers is number 1. Also for every non-zero number a the inverse of a under multiplication is $a^{-1} = \frac{1}{a}$. Therefore we see that $(\mathbf{R} \setminus \{0\}, \cdot)$ is an abelian group. Another example is $(\mathbf{R}^+ \setminus \{0\}, \cdot)$. However neither $(\mathbf{Z} \setminus \{0\}, \cdot)$ nor $(\mathbf{N} \setminus \{0\}, \cdot)$ are groups, because for instance $\frac{1}{2} \notin \mathbf{Z}$. As we will see in the sequel, there are examples of groups, when it is more natural to denote the group operation by $\cdot$. In that case we denote the identity element by 1 and the inverse element of a by a^{-1}. We then speak of *multiplicative groups* as being in contrast to additive groups, when the group operation is denoted by $+$, the identity element by 0 and the inverse of a by $-a$. However, the reader should remember the there is no difference between the additive and the multiplicative group other than just notation.

Typically we simplify the expression of multiplication and write ab instead of $a \cdot b$.

In case of numbers, there is a useful property which involves both addition and multiplication. Namely, for any numbers a, b, c

$$a \cdot (b + c) = a \cdot b + a \cdot c. \tag{13.10}$$

Given a set F and operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ we say that the operation $\cdot$ *distributes* over the operation $+$ if for any $x, y, z \in F$ property (13.10) is satisfied.

We can summarize the properties of addition and multiplication of real numbers by saying that $(\mathbf{R}, +)$ and $(\mathbf{R} \setminus \{0\}, \cdot)$ are abelian groups and the multiplication distributes over addition.

Definition 13.22 A *field* is a triple $(F, +, \cdot)$, where $+ : F \times F \to F$ and $\cdot : F \times F \to F$ are operations in F such that $(F, +)$ and $(F \setminus \{0\}, \cdot)$ are abelian groups and $\cdot$ is distributive over $+$.

As in the case of groups, if the operations are clear from the context, we simply say that F is a field.

Example 13.23 The set of complex numbers $\mathbf{C}$ and the set of rational numbers $\mathbf{Q}$ are fields.

Example 13.24 The integers $\mathbf{Z}$ do not form a field. In particular, $2 \in \mathbf{Z}$, but $2^{-1} = \frac{1}{2} \notin \mathbf{Z}$.

Example 13.25 A very useful field is $\mathbf{Z}_2$, the set of integers modulo 2. The rules for addition and multiplication are as follows:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

We leave it to the reader to check that $(\mathbf{Z}_2, +, \cdot)$ is a field.

Example 13.26 Another field is $\mathbf{Z}_3$, the set of integers modulo 3. The rules for addition and multiplication are as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Again, we leave it to the reader to check that $(\mathbf{Z}_3, +, \cdot)$ is a field. However, we note that the inverse of 1 under addition in $\mathbf{Z}^3$ is 2 and the inverse of 2 under multiplication in $\mathbf{Z}_3$ is 2.

Example 13.27 $\mathbf{Z}_4$, the set of integers modulo 4 is not a field. The rules for addition and multiplication are as follows:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Observe that the element 2 does not have an inverse with respect to multiplication in $\mathbf{Z}_4$.

13.2.2 Vector Spaces

Let $(G, +)$ be a group. We have an operation $\mathbf{Z} \times G \to G$, which assigns mg to $m \in \mathbf{Z}$ and $g \in G$. Such an operation is an example of an *external operation*, because the first operand is not from the group G. In the case of real vectors such an external operation makes sense even if the first operand is an arbitrary real number. More precisely, given a real vector $\mathbf{x} := (x_1, x_2, \dots x_d) \in \mathbf{R}^n$ and a real number $a \in \mathbf{R}$ we define

$$a\mathbf{x} := (ax_1, ax_2, \dots ax_n). \tag{13.11}$$

In this case, or more generally when the first operand is from a field, the external operation is referred to as a *scalar multiplication* and the first operand is called a *scalar*. One easily verifies that when $a \in \mathbf{Z}$ then the definition (13.11) coincides with (13.6).

The group $(\mathbf{R}^n, +)$ considered together with the scalar multiplication $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ given by (13.11) is an example of a *vector space* over the field $\mathbf{R}$. An example we are most familiar with is $\mathbf{R}^3$, because it is used as coordinate space to describe our everyday world.

In the general definition below the field may be arbitrary.

Definition 13.28 A *vector space* over a field F is a set V with two operations, vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ such that $(V, +)$ is an abelian group and the scalar multiplication satisfies the following rules:

1. For every $v \in V$, 1 times v equals v where $1 \in F$ is the unique element one in the field.
2. For every $v \in V$ and $\alpha, \beta \in F$
$$\alpha(\beta v) = (\alpha\beta)v$$
3. For every $\alpha \in F$ and all $u, v \in V$,
$$\alpha(u + v) = \alpha u + \alpha v.$$
4. For all $\alpha, \beta \in F$ and every $v \in V$
$$(\alpha + \beta)v = \alpha v + \beta v.$$

Example 13.29 Let F be a field and $n > 0$ an integer. For $(x_1, x_2, \dots x_n) \in F^n$, $(y_1, y_2, \dots y_n) \in F^n$ and $\alpha \in F$ define

$$(x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n) := (x_1 + y_1, x_2 + y_2, \dots x_n + y_n) \tag{13.12}$$

$$\alpha(x_1, x_2, \dots x_n) := (\alpha x_1, \alpha x_2, \dots \alpha x_n). \tag{13.13}$$

It is straightforward to verify that F^n with these two operations is a vector space over the field F.

From the point of view of this book the important case are vector spaces $\mathbf{Z}_2^n$ over the finite field $\mathbf{Z}_2$. Such a vector space is finite itself. For instance

$$\mathbf{Z}_2^3 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

consists of eight vectors.

We already know that the operation given by (13.12) may also be considered in $\mathbf{Z}^n$ and $\mathbf{Z}^n$ with this operation is an abelian group. We also know that the multiplication of an integer vector by an integer makes sense and it is just a shorthand for adding a number of copies of a vector. However, we should be aware that $\mathbf{Z}^n$ is not a vector space, because $\mathbf{Z}$ is not a field. Nevertheless there are some similarities of the group $\mathbf{Z}^n$ and a vector space. Since the group $\mathbf{Z}^n$ is very important from the point of view of applications to homology, in the sequel we will study these similarities as well as differences.

13.2.3 Linear Combinations and Bases

Let V be a vector space over a field F. Given k vectors $w_1, w_2, \ldots, w_k \in V$ and k scalars $\alpha_1, \alpha_2, \ldots \alpha_k \in F$ we can form the expression

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_k w_k,$$

called the *linear combination* of vectors $w_1, w_2, \ldots, w_k$ with *coefficients* α_1, α_2, $\ldots \alpha_k$.

Slightly more general is the *linear combination of a family of vectors* $\{w_j\}_{j\in J} \subset V$ with coefficients $\alpha_1, \alpha_2, \ldots \alpha_k \in F$, which has the form

$$\sum_{j\in J} \alpha_j w_j. \tag{13.14}$$

Of course in order to guarantee that such a linear combination makes sense we need to assume that for all but a finite number of $j \in J$ the coefficients $\alpha_j = 0$. In the sequel, whenever we write a linear combination of the form (13.14) we assume that all but a finite number of coefficients are zero.

The following concept is fundamental in the study of vector spaces.

Definition 13.30 A family $\{w_j\}_{j\in J} \subset V$ of vectors in a vector space V is a *basis* of V if for every vector $v \in V$ there exists a unique family of scalars $\{\alpha_j\}_{j\in J} \subset F$ such that all but a finite number of them are zero and

$$v = \sum_{j\in J} \alpha_j w_j.$$

One can prove that in every vector space there exists a basis, although the proof is not easy (see ....). A vector space V is called *finitely dimensional* if there exists a basis which is finite. In a finitely dimensional vector space any two bases have the same number of elements (see ...). This number is then called the *dimension* of the vector space.

Assume V is a finitely dimensional vector space over a field F and

$$\{w_1, w_2, \dots w_n\} \subset V$$

is a basis. Then for every $v \in V$ there exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$v = \alpha_1 w_1 + \alpha_2 w_2 + \dots \alpha_n w_n.$$

They are called the *coordinates* of v in the basis $\{w_1, w_2, \dots w_n\}$. These coordinates form a vector $(\alpha_1, \alpha_2, \dots, \alpha_n) \in F^n$, called the *coordinate vector* of v.

In an arbitrary vector space there is in general no natural way to select a basis. There is, however, an important special case where there is a natural choice of basis, namely, the vector space F^n. The *canonical basis* of F^n consists of vectors $\{\mathbf{e}_1^n, \mathbf{e}_2^n, \dots \mathbf{e}_n^n\}$, where

$$\mathbf{e}_i^n := (0, 0, \dots 0, 1, 0, \dots 0)$$

has all coordinates zero except the i-th coordinate, which is one. To see that this is a basis observe that for every $\mathbf{x} = (x_1, x_2, \dots x_n) \in F^n$

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

and the uniqueness of this decomposition is obvious.

This basis will be used throughout the book. We will usually drop the superscripts n in $\mathbf{e}_i^n$ if it is clear from the context what n is.

To see why the canonical basis is a natural choice observe that coordinate vector of the element $\mathbf{x} = (x_1, x_2, \dots x_n) \in F^n$ in this basis is $(x_1, x_2, \dots x_n)$, i.e. it is indistinguishable from the element $\mathbf{x}$ itself.

There are two other important concepts of linear algebra based on linear combinations: linear independence and spanning.

Definition 13.31 Let V be a vector space over a field F. A set of vectors $\{v_j\}_{j\in J} \subset V$ is *linearly independent* if the only way of writing 0 as a linear combination of vectors $\{v_j\}_{j\in J}$ is by taking all coefficients equal to zero, i.e. if for any $\{\alpha_j\}_{j\in J} \subset F$

$$\sum_{j\in J} \alpha_j v_j = 0 \;\Rightarrow\; \alpha_j = 0 \text{ for all } j \in J.$$

The set of vectors $\{v_j\}_{j\in J}$ *spans* V if every element $w \in V$ can be written (not necessarily in a unique way) as a linear combination of vectors in $\{v_j\}_{j\in J}$, i.e. if

$$w = \sum_{j\in J} \alpha_j v_j$$

for some $\{\alpha_j\}_{j\in J} \subset F$ such that all but a finite number of them are zero.

The following theorem is a convenient criterion for a set to be a basis.

Theorem 13.32 *A set of vectors* $\{v_j\}_{j\in J} \subset V$ *is a basis if and only if it is linearly independent and spans* V.

Proof. Let $\{v_j\}_{j\in V} \subset V$ be a basis in a vector space V. Then obviously every vector is a linear combination of elements in $\{v_j\}_{j\in J}$. Therefore the set $\{v_j\}_{j\in J}$ spans V. Its linear independence follows from the uniqueness of decomposition of $0 \in V$.

Conversely, if $\{v_j\}_{j\in J}$ is linearly independent and spans V then every element $w \in V$ equals $\sum_{j\in J} \alpha_j v_j$ for some coefficients $\{\alpha_j\}_{j\in J}$. Assume that also $v = \sum_{j\in J} \beta_j v_j$. Then

$$0 = \sum_{j\in J} (\alpha_j - \beta_j) v_j$$

and from the linear independence of $\{v_j\}_{j\in J}$ it follows that $\alpha_j - \beta_j = 0$, i.e. $\alpha_j = \beta_j$ for all $j \in J$. We conclude that the representation $w = \sum_{j\in J} \alpha_j v_j$ is unique. □

The following theorem is a very convenient characterization of bases in vector spaces.

Theorem 13.33 *(see ....) Let* V *be a vector space. A set* $\{w_j\}_{j\in J} \subset V$ *is a basis of* V *if and only if it is a maximal linearly independent set. Also* $\{w_j\}_{j\in J} \subset V$ *is a basis of* V *if and only if it is a minimal set spanning* V.

Exercises

13.7 (a) Write down the tables of addition and multiplication for $\mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_8$.
(b) If $\mathbf{Z}'_n := \mathbf{Z}_n \setminus 0$, show that $\mathbf{Z}'_5$ is a multiplicative group but $\mathbf{Z}'_6, \mathbf{Z}'_8$ are not.
(c) Let now $\mathbf{Z}^*_n := \{k \in \mathbf{Z}_n : k \text{ and } n \text{ are relatively prime}\}$. Show that $\mathbf{Z}^*_n$ is a multiplicative group for any positive integer n.

13.8 Determine the orders of all elements of $\mathbf{Z}^*_5, \mathbf{Z}^*_6, \mathbf{Z}^*_8$, where$\mathbf{Z}^*_n$ is defined in the preceding exercise.

13.9 Prove that the set of rational numbers $\mathbf{Q}$ is a field.

13.10 Let $\mathbf{Z}_n$ denote the set of integers modulo n. For which n is $\mathbf{Z}_n$ a field?

13.11 Prove the following statements.

(a) If G is a finite multiplicative group and $a \in G$, then $a^{|G|} = 1$.
(**Hint:** Use Proposition 2.4 with $H = \langle a \rangle$)
(b) (Fermat's Little Theorem) If p is a prime and p does not divide $a \in \mathbf{Z}$ then $a^{p-1} \equiv 1 \pmod{p}$.
(**Hint:** Recall Exercise 2(c) Section 1)
(c) If p is a prime then $b^p \equiv b \pmod{p}$ for all $b \in \mathbf{Z}$.

13.3 Homomorphisms

If we wish to compare two groups or two vector spaces, then we need to be able to talk about functions between them. Of course these functions need to preserve the operations.

13.3.1 Homomorphisms of Groups

In the case of groups this leads to the following definition.

Definition 13.34 *Let G and G' be two abelian groups.* A map $f : G \to G'$ is called a *homomorphism* if

$$f(g_1 + g_2) = f(g_1) + f(g_2)$$

for all $g_1, g_2 \in G$.

There are some immediate consequences of this definition. For example, as the following argument shows, homomorphisms map the identity element to the identity element.

$$\begin{aligned} f(0) &= f(0+0) = f(0) + f(0) \\ f(0) - f(0) &= f(0) \\ 0 &= f(0) \end{aligned}$$

A trivial induction argument shows that

$$f(mg) = mf(g)$$

for all $m \in \mathbf{Z}$ and $g \in G$. As a consequence we obtain

$$f(m_1 g_1 + m_2 g_2) = m_1 f(g_1) + m_2 f(g_2)$$

for any $m_1, m_2 \in \mathbf{Z}$ and $g_1, g_2 \in G$.

Proposition 13.35 *Let $f : G \to G'$ be a homomorphism of groups. Then*

(a) for any subgroup H of G, its image $f(H)$ is a subgroup of G';
(b) for any subgroup H' of G', its inverse image $f^{-1}(H')$ is a subgroup of G;
(c) if f is bijective (i.e. one-to-one and onto) then its inverse $f^{-1} : G' \to G$ also is a bijective homomorphism.

Proof. (a) Let $f(h_1), f(h_2)$ be two elements of $f(H)$. Then

$$f(h_1) - f(h_2) = f(h_1 - h_2) \in f(H).$$

Therefore by Proposition 13.8 $f(H)$ is a subgroup of G'.

(b) Let $g_1, g_2 \in f^{-1}(H')$. Then

$$f(g_1 - g_2) = f(g_1) - f(g_2) \in H'.$$

Therefore $g_1 - g_2 \in f^{-1}(H')$ and again by Proposition 13.8 $f^{-1}(H')$ is a subgroup of G.

c) follows from similar types of arguments and is left to the reader. □

Definition 13.36 The set $\operatorname{im} f := f(G)$ is called the *image* or *range* of f in G'. By the previous proposition it is a subgroup of G'. The set

$$\ker f := f^{-1}(0) = \{a \in G \mid f(a) = 0\}$$

is called the *kernel* of f. Again by the previous proposition it is a subgroup of G.

Definition 13.37 A homomorphism $f : G \to G'$ is called an *epimorphism* if it is surjective (or onto) i.e. $\operatorname{im} f = G'$ and a *monomorphism* if it is injective (or 1-1), i.e. for any $a \neq b$ in G, $f(a) \neq f(b)$. This later condition obviously is equivalent to the condition $\ker f = 0$. Finally, f is called an *isomorphism* if it is both a monomorphism and an epimorphism.

The last definition requires some discussion since the word isomorphism takes different meanings in different branches of mathematics. Let X, Y be any sets and $f : X \to Y$ any map. Then f is called *invertible* if there exists a map $g : Y \to X$, called *inverse* of f with the property

$$gf = \operatorname{id}_X \text{ and } fg = \operatorname{id}_Y \tag{13.15}$$

where id_X and id_Y denote the identity maps on X and Y respectively. It is easy to show that f is invertible if and only if it is bijective. If this is the case, g is uniquely determined and denoted by f^{-1}. When we speak about a particular class of maps, by an invertible map or an isomorphism we mean a map which has an inverse in the same class of maps. For example, if continuous maps are of concern, an isomorphism would be a continuous map which has a continuous inverse. The continuity of a bijective map does not guarantee, in general, the continuity of its inverse. Proposition 3.1(c) guarantees that this problem does not occur in the class of homomorphisms. Thus, a homomorphism is an isomorphism if and only if it is invertible in the class of homomorphisms.

When $G = G'$, a homomorphism $f : G \to G$ is also called an *endomorphism* and an isomorphism $f : G \to G$ is called an *automorphism*.

Groups G and G' are called *isomorphic*, if there exists an isomorphism $f : G \to G'$. We then write $f : G \xrightarrow{\cong} G'$ or just $G \cong G'$ if f is irrelevant. It is easy to see that $G \cong G'$ is an equivalence relation. We shall often permit ourselves to identify isomorphic groups.

Example 13.38 Let G be a cyclic group of infinite order generated by a. Then $f : \mathbf{Z} \to G$ defined by $f(n) = na$ is an isomorphism with the inverse defined by $f^{-1}(na) = n$. By the same argument, any cyclic group of order k is isomorphic to $\mathbf{Z}_k$.

Example 13.39 Let H be a subgroup of G and define $q : G \to G/H$ by the formula $q(a) := a + H$. It is easy to see that q is an epimorphism and its kernel is precisely H. This map is called the *canonical quotient homomorphism*.

Example 13.40 Let A, B be subgroups of G such that $G = A \oplus B$. Then the map $f : A \times B \to G$ defined by $f(a, b) = a + b$ is an isomorphism with the inverse defined by $f^{-1}(c) = (a, b)$ where $c = a + b$ is the unique decomposition of $c \in G$ with $a \in A$ and $b \in B$. This can be generalized to direct sums and products of any finite number of groups.

Example 13.41 Let A, B, and G be as in Example 13.40. The inclusion map $i : A \to G$ is a monomorphism and the projection map $p : G \to A$ defined by $p(c) = a$ where $c = a + b$ with $a \in A$ and $b \in B$, is an epimorphism. Note that $pi = \mathrm{id}_A$ hence p may be called a *left inverse* of i and i a *right inverse* of p. Note that a left inverse is not necessarily unique. Indeed, take subgroups $A = \mathbf{Z}\mathbf{e}_1, B = \mathbf{Z}\mathbf{e}_2$ of $\mathbf{Z}^2$. Another choice of a left inverse of i is $p'(n\mathbf{e}_1 + m\mathbf{e}_2) = (n + m)\mathbf{e}_1$ (a "slanted" projection).

We leave as an exercise the following useful result.

Proposition 13.42 *If $f : G \to G$ is a group homomorphism, then $f(T(G)) \subset T(G)$ where $T(G)$ is the torsion subgroup of G.*

Let now $f : G \to G'$ be a homomorphism and $H = \ker f$. Then, for any $a \in G$ and $h \in H$, we have $f(a + h) = f(a)$. Hence the image of any coset $a + H$ under f is

$$f(a + H) = \{f(a)\} .$$

Moreover, that image is independent of the choice of a representative of a coset $a + H$. Indeed, if $a + H = b + H$ then $b - a \in H$ thus $f(b) = f(a)$. We may now state the following

Theorem 13.43 *Let $f : G \to G'$ be a homomorphism and $H = \ker f$. Then the map*

$$\bar{f} : G/H \to \operatorname{im} f$$

defined by $\bar{f}(a + H) = f(a)$ is an isomorphism, called the quotient isomorphism.

Proof. By the preceding discussion, the formula for $\bar{f}$ is independent of the choice of coset representatives, thus $\bar{f}$ is well defined. Since,

$$\bar{f}((a + H) + (b + H)) = \bar{f}(a + b + H) = f(a + b) = f(a) + f(b)$$

it is a homomorphism. $\bar{f}$ is a monomorphism since $\bar{f}(a + H) = 0$ implies $f(a) = 0$, i.e. $a \in \ker f = H$ or $a + H = H$.

Finally, $\bar{f}$ is also an epimorphism since $\operatorname{im} \bar{f} = \operatorname{im} f$. □

As a corollary we obtain the following special case known as the Fundamental Epimorphism Theorem in group theory.

Corollary 13.44 *If $f : G \to G'$ is an epimorphism then*

$$\bar{f} : G/\ker f \to G'$$

is an isomorphism.

Example 13.45 Let $q : G \to G/H$ be the canonical homomorphism from Example 13.39. Then $\bar{q} = \mathrm{id}_{G/H}$, so this is the trivial case of Theorem 3.1.

Example 13.46 Let $f : \mathbf{Z} \to \mathbf{Z}_n$ be given by $f(a) = a \bmod n$. Then f is a well defined epimorphism with $\ker f = n\mathbf{Z}$. Thus $\bar{f} : \mathbf{Z}/n\mathbf{Z} \stackrel{\cong}{\to} \mathbf{Z}_n$.

Example 13.47 Let's go back to p' in Example 13.41. $\mathrm{im}\, p' = \mathbf{Zi} = A$ and $\ker p' = \mathbf{Z}(\mathbf{e}_2 - \mathbf{e}_1)$ Thus $\bar{f} : \mathbf{Z}^2/\mathbf{Z}(\mathbf{e}_2 - \mathbf{e}_1) \stackrel{\cong}{\to} \mathbf{Zi}$. Note that $\mathbf{Z}^2 = \mathbf{Ze}_1 \oplus \mathbf{Z}(\mathbf{e}_2 - \mathbf{e}_1) = \mathrm{im}\, p' \oplus \ker p'$. This observation will be later generalized.

Example 13.48 Consider Example 13.20 in terms of the quotient isomorphism. Let S^1 be the unit circle in the complex plane, i.e. the set defined by $|z| = 1,\ z = x + iy \in \mathbf{C}$, i the primitive square root of -1. Then S^1 is a multiplicative group with the complex number multiplication and the unity $1 = 1 + i0$. We define $f : \mathbf{R} \to S^1$ by $f(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$. Then f is a homomorphism from the additive group of $\mathbf{R}$ to the multiplicative group S^1. It is an epimorphism with the kernel $\ker f = 2\pi\mathbf{Z}$. Thus $\bar{f} : \mathbf{R}/2\pi\mathbf{Z} \stackrel{\cong}{\to} S^1$.

Theorem 13.49 *Assume $G_1, G_2, \ldots G_n$ and $H_1, H_2, \ldots H_n$ are groups and $H_i \subset G_i$ are subgroups. Then*

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n/H_1 \oplus H_2 \oplus \cdots \oplus H_n \cong G_1/H_1 \oplus G_2/H_2 \oplus \cdots \oplus G_n/H_n$$

Proof. Let $\pi_i : G_i \to G_i/H_i$ denote the quotient homomorphism. Then we have an epimorphism $\pi : G_1 \oplus G_2 \oplus \cdots \oplus G_n \to G_1/H_1 \oplus G_2/H_2 \oplus \cdots \oplus G_n/H_n$ defined by

$$\pi(g_1 + g_2 + \cdots + g_n) := [g_1] + [g_2] + \cdots + [g_n].$$

It is straightforward to verify that $\ker \pi = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Therefore the conclusion follows from Corollary 13.44. □

13.3.2 Linear Maps

In the case of a vector space V over a field F we have two operations: vector addition $+$ which makes $(V, +)$ an abelian group and scalar multiplication. The definition of a map preserving the structure of a vector space must take care of both of them. Therefore we have the following definition.

Definition 13.50 *Assume V, V' are two vector spaces over the same field F.* A map $f : V \to V'$ is called a *linear map* if

$$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$$

for all $\alpha_1, \alpha_2 \in F$ and $v_1, v_2 \in V$.

The terminology "linear map" comes from the fact that the graph of a linear map of $f : \mathbf{R}^1 \to \mathbf{R}^1$ is a line.

We will study only linear maps of finitely dimensional vector spaces.

Let V and V' be two finitely dimensional vector spaces with bases, respectively, $W := \{w_1, w_2, \ldots, w_n\}$ and $W' := \{w'_1, w'_2, \ldots, w'_m\}$. Let $f : V \to V'$ be a linear map and let $v = \sum_{j=1}^{n} x_j w_j$ be an element of V. Then

$$f(v) = f\left(\sum_{j=1}^{n} x_j w_j\right) = \sum_{j=1}^{n} x_j f(w_j). \tag{13.16}$$

This means that f is uniquely determined by its values on the elements of a basis in V. In particular, in order to define a linear map it is enough to define its values on elements of a selected basis of V.

On the other hand, the values $f(w_j)$ may be written as linear combinations of the elements of the basis in V'

$$f(w_j) = \sum_{i=1}^{m} a_{ij} w'_i \,. \tag{13.17}$$

From (13.16) and (13.17) we obtain

$$f(v) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) w'_i \,. \tag{13.18}$$

Therefore, the coordinates $x'_1, x'_2, \ldots, x'_m$ of $f(v)$ in the basis W' are described by the following system of linear equations

$$\begin{array}{lllll} x'_1 = a_{11}x_1 & +a_{12}x_2 & +\ldots+ & a_{1n}x_n \\ x'_2 = a_{21}x_1 & +a_{22}x_2 & +\ldots+ & a_{2n}x_n \\ \ldots \quad \ldots & \ldots & \ldots & \ldots \\ x'_m = a_{m1}x_1 & +a_{m2}x_2 & +\ldots+ & a_{mn}x_n \end{array} \tag{13.19}$$

13.3.3 Matrix Algebra

In the study of systems of linear equations such as (13.19) it is convenient to use the language of matrices. Recall that an *m by n matrix* with *entries a_{ij}* in the field F is an array of elements of F of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{13.20}$$

The number m is referred to as the number of rows and n as the number of columns of the matrix. When these numbers are clear from the context, we often use a compact notation $[a_{ij}]$. Even more compact is the convention of denoting the whole matrix by one capital letter. Usually the entries are denoted by lower case letters and the whole matrix by the corresponding upper case letter, so in the case of (13.20) we write $A = [a_{ij}]$. We will denote the set of matrices with m rows and n columns by $\mathrm{M}_{m,n}(F)$, where F is the field form which the entries of the matrix are taken.

We define the multiplication of matrices as follows. If $B = [b_{jk}] \in \mathrm{M}_{n,p}(F)$ is another matrix whose number of columns is the same as the number of rows of A, then the em product AB is the matrix $C = [c_{ik}] \in \mathrm{M}_{m,p}(F)$ whose entries are defined by

$$c_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}.$$

It should be emphasized that matrix multiplication is in general not commutative.

Using matrix notation, equation (13.19) may be written as

$$\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{13.21}$$

The *identity matrix*

$$I_{n\times n} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

with n rows and columns has all entries zero except the diagonal entries, which are one. It is straightforward to verify that the identity matrix is an identity element for matrix multiplication, i.e.

$$AI_{n\times n} = A \quad \text{and} \quad I_{n\times n}B = B$$

for $A \in \mathrm{M}_{m,n}(F)$ and $B \in \mathrm{M}_{n,p}(F)$.

The matrices whose number of rows is the same as the number of columns are called *square matrices*. A square matrix $A \in \mathrm{M}_{n,n}(F)$ is *invertible* or *nonsingular* if there exists a matrix $B \in \mathrm{M}_{n,n}(F)$ such that

$$AB = BA = I_{n\times n}.$$

One easily verifies that if such a matrix B exists, then it is unique. It is called the *inverse of matrix* A and it is denoted by A^{-1}.

If $A_1, A_2, \ldots A_k$ are matrices with the same number of rows then the matrix build of all columns of A_1 followed by all columns of A_2 and so on up to the last matrix will be denoted by

$$[\, A_1 \; A_2 \; \dots \; A_k \,] \,.$$

Sometimes, to emphasize the process in which the matrix was obtained, we will put vertical bars between the matrices A_i, i.e. we will write

$$[\, A_1 \mid A_2 \mid \; \dots \; \mid A_k \,] \,.$$

The *transpose* of a matrix $A = [a_{ij}] \in \mathrm{M}_{m,n}(F)$ is the matrix

$$A^T := [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} .$$

We will identify an element $\mathbf{x} = (x_1, x_2, \dots, x_n) \in F^n$ with the row matrix $[x_1, x_2, \dots, x_n]$.

Using the introduced notation equation (13.19) becomes

$$(\mathbf{x}')^T = A\mathbf{x}^T.$$

In situations when this does not lead to confusion we will allow ourselves the freedom of identifying the vector $\mathbf{x}$ with its transpose $\mathbf{x}^T$ to reduce the amount of transpose signs. With this convention equation (13.19) becomes just

$$\mathbf{x}' = A\mathbf{x}. \tag{13.22}$$

Going back to the linear map $f : V \to V'$ let $\mathbf{x} := (x_1, x_2, \dots x_n)$ be the vector of coordinates of $v \in V$ in the basis W and let $\mathbf{x}' := (x_1', x_2', \dots x_n')$ be the vector of coordinates of $f(v) \in V'$ in the basis W'. Then the equation (13.19) describing the relation between the coordinates of v and $f(v)$ may be written in the compact form as (13.22), where the matrix $A = [a_{ij}]$ consists of coefficients in (13.18). This matrix is referred to as the *matrix of f in the bases* W and W'.

Conversely, if $A = [a_{ij}]$ is an $m \times n$ matrix, then the formula (13.18) defines a unique linear map $f : V \to V'$.

If $f : F^n \to F^m$ is a linear map, then its matrix in the canonical bases is simply called *the matrix of f*. Since this matrix is frequently needed, we will use A_f as the notation for it.

Note that in the case of a homomorphism $f : F^n \to F^m$ the equation

$$\mathbf{y} = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in F^n, \mathbf{y} \in F^m$$

may be rewritten using matrix notation as

$$\mathbf{y} = A_f \mathbf{x},$$

because the elements $\mathbf{x}, \mathbf{y} \in F^n$ are indistinguishable from their coordinates in the canonical bases. Nevertheless, we want to have a clear distinction between the matrix A, which is an array of numbers, and the linear map defined by this matrix.

An elementary calculation shows that if $f' : F^m \to F^k$ is another linear map then

$$A_{f'f} = A_{f'} A_f. \tag{13.23}$$

Actually, this fact is the motivation for the definition of the product of two matrices.

Let $\mathrm{id}_{F^n} : F^n \to F^n$ be the identity map. It is straightforward to verify that id_{F^n} is an isomorphism and its matrix is $I_{n\times n}$.

Therefore it follows from (13.30) that if $f : F^n \to F^n$ is an isomorphism, then the matrix A_f is nonsingular and

$$A_f^{-1} = A_{f^{-1}}.$$

Exercises

13.12 Show that $\mathbf{Z}_6 \cong \mathbf{Z}_2 \times \mathbf{Z}_3$.

13.13 If m and n are relatively prime, show that $\mathbf{Z}_m \oplus \mathbf{Z}_n \simeq \mathbf{Z}_{mn}$ (see Exercise 13.7).

13.14 Prove Proposition 13.42.

13.4 Free Abelian Groups

13.4.1 Bases in Groups

Replacing vectors by group elements and arbitrary field scalars by integers we can carry over the concept of a linear combination from vector spaces to groups. More precisely, given an abelian group G, the linear combination of $g_1, g_2, \dots g_k \in G$ with coefficients $m_1, m_2, \dots m_k \in \mathbf{Z}$ is

$$m_1 g_1 + m_2 g_2 + \cdots + m_k g_k.$$

Similarly, if $\{g_j\}_{j\in J} \subset G$ and $\{m_j\}_{j\in J} \subset \mathbf{Z}$, the linear combination

$$\sum_{j\in J} m_j g_j$$

makes sense whenever all but a finite number of $m_j = 0$.

A basis in an abelian group may be defined exactly in the same way as in a vector space.

Definition 13.51 A family $\{g_j\}_{j\in J} \subset G$ of elements of a group G is a *basis* of G if for every $h \in G$ there exists a unique family of integers $\{m_j\}_{j\in J} \subset \mathbf{Z}$ such that all but a finite number of them are zero and

$$h = \sum_{j\in J} m_j g_j .$$

There is however a fundamental difference between vector spaces and groups: in every vector space there exists a basis but this is not true for every group.

Definition 13.52 A group G which has a basis is called *free*.

Example 13.53 Consider the group $\mathbf{Z}^n$. Since it contains the canonical vectors $\{\mathbf{e}_1^n, \mathbf{e}_2^n, \ldots \mathbf{e}_n^n\}$, it is natural to ask if this is a basis for $\mathbf{Z}^n$. Actually, the verification is straightforward. Therefore $\mathbf{Z}^n$ is a free group.

To see that not every group possesses a basis consider the following example.

Example 13.54 The group $\mathbf{Z}_2$ of integers modulo 2 is not free. To prove this observe that the basis cannot contain 1, because

$$0 = 0 \cdot 1 = 2 \cdot 1,$$

i.e. the uniqueness of decomposition is violated. Obviously 0 never belongs to a basis, neither in a vector space nor in a group, because any multiplicity of zero is zero, which again violates the uniqueness of decomposition. Therefore there is no basis in $\mathbf{Z}_2$, i.e. $\mathbf{Z}_2$ is not free.

Example 13.55 The group of rational numbers $\mathbf{Q}$ is not free. To see this assume that $\{g_j\}_{j\in J}$ forms a basis for $\mathbf{Q}$. Recall that any element $a \in Q$ can be written in the form $a = p/q$ where p and q are relatively prime integers. Assume first that the basis consists of a unique element $g = p/q$. Then $g/2 \in Q$, but given an integer n it is impossible to solve the equation $ng = g/2$. Therefore, the basis must contain more than one element. In particular, there exists p_1/q_1 and p_2/q_2 in the basis. Now observe that

$$(p_2 q_1) p_1/q_1 = p_1 p_2 = (p_1 q_2) p_2/q_2$$

which violates the uniqueness condition.

Example 13.56
Let G be a free abelian group with a basis $\{g_1, g_2, \ldots, g_n\}$. By the definition of a basis,

$$G = \mathbf{Z}g_1 \oplus \mathbf{Z}g_2 \oplus \cdots \oplus \mathbf{Z}g_n \ .$$

Example 13.57 Consider the group $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$. Then $\mathbf{Z}^2 = \mathbf{Z}\mathbf{e}_1 \oplus \mathbf{Z}\mathbf{e}_2$. This decomposition of $\mathbf{Z}^2$ to a direct sum is related to a particular choice of basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. As in vector spaces, there may be many bases, and hence, many direct sum decompositions, e.g. $\mathbf{Z}^2 = \mathbf{Z}\mathbf{e}_1 \oplus \mathbf{Z}(\mathbf{e}_2 - \mathbf{e}_1)$.

The concepts of linear independence and spanning may also be carried over from vector spaces to groups. The analogous definitions for free abelian groups are very similar. For historical reasons, in the case of groups we usually say the a set generates the group rather than it spans it.

Definition 13.58 Let G be an abelian group G. A set of group elements $\{g_j\}_{j \in J} \subset G$ is *linearly independent* if the only way of writing 0 as a linear combination of elements $\{g_j\}_{j \in J}$ is by taking all coefficients equal to zero, i.e. if for any $\{m_j\}_{j \in J} \subset \mathbf{Z}$

$$\sum_{j \in J} m_j g_j = 0 \;\Rightarrow\; m_j = 0 \text{ for all } j \in J.$$

The set of group elements $\{g_1, g_2, \ldots g_k\}$ *generates* G if every element $h \in G$ can be written as

$$h = \sum_{j \in J} m_j g_j$$

for some $\{m_j\}_{j_J} \subset \mathbf{Z}$.

The following theorem is a group counterpart of Theorem 13.32 for vector spaces and its proof is analogous.

Theorem 13.59 *A set of group elements $\{g_j\}_{j \in J} \subset G$ is a basis if and only if it is linearly independent and generates G.*

However, in the case of free groups there is no counterpart of Theorem 13.33 as the following example shows.

Example 13.60 The group of integers $\mathbf{Z}$ is a free abelian group generated by a basis consisting of a single element: either $\{1\}$ or $\{-1\}$. Observe that for any $k \in \mathbf{Z} \setminus \{0\}$, $\{k\}$ is a maximal linearly independent set. Indeed, for any $l \in \mathbf{Z}$ we have

$$lk + (-k)l = 0,$$

which shows that any superfamily of $\{k\}$ is not linearly independent. However, if $k \neq \pm 1$, then $\{k\}$ does not generate $\mathbf{Z}$. This is easily seen by noting that if $\{k\}$ did generate $\mathbf{Z}$, then there would be an integer n such that $nk = 1$.

Despite this example it is still possible to generalize a part of Theorem 13.33 to groups.

Lemma 13.61 *Let G be an abelian group. Assume*

$$\{g_j\}_{j\in J} \subset \{g_j\}_{j\in \bar{J}} \subset G$$

are two families of group elements such that the first generates G and the other is linearly independent. Then the families are equal.

Proof. Assume the contrary and select $j_0 \in \bar{J} \setminus J$. Since $\{g_j\}_{j\in J}$ generates G, we have

$$g_{j_0} = \sum_{j\in \bar{J}} m_j g_j$$

for some integers $\left\{m_j \mid j \in \bar{J}\right\}$. But then

$$g_{j_0} - \sum_{j\in \bar{J}} m_j g_j = 0,$$

which contradicts the fact that $\{g_j\}_{j\in \bar{J}}$ is linearly independent. □

Corollary 13.62 *A basis of an abelian group is a maximal linearly independent set and a minimal set generating G.*

However, as the following example shows, the converse statement in the case of groups is not true.

Example 13.63 Consider again the group $\mathbf{Z}^2$. The set $\{2\mathbf{e}_1, 3\mathbf{e}_2\}$ is not a basis for $\mathbf{Z}^2$ even though it is a maximal linearly independent set in $\mathbf{Z}^2$. This set is a basis for $2\mathbf{Z} \times 3\mathbf{Z}$ which is a proper subgroup of $\mathbf{Z}^2$ of the same rank 2.

If there is a finite set generating a group, then it is called a *finitely generated group.*

Theorem 13.64 *Any two bases of a finitely generated free abelian group G are finite and have the same number of elements. This number is called the* rank *of G.*

Proof. Assume $h_1, h_2, \ldots h_n$ generate G and $\{g_j\}_{j\in J}$ is a basis of G. Then

$$h_i = \sum_{j\in J} m_{ij} g_j$$

for some coefficients $\{m_{ij} \mid i = 1, 2, \ldots n, j \in J\}$. Since for any $i = 1, 2, \ldots n$ the set

$$\{j \in J \mid m_{ij} \neq 0\}$$

is finite, the set

$$\bar{J} := \bigcup_{i=1}^{n} \{j \in J \mid m_{ij} \neq 0\}$$

is also finite and

$$h_i = \sum_{j \in \bar{J}} m_{ij} g_j .$$

It follows that for every $h \in G$

$$h = \sum_{i=1}^{n} k_i h_i = \sum_{i=1}^{n} \sum_{j \in \bar{J}} k_i m_{ij} g_j$$

i.e. $\{g_j\}_{j \in \bar{J}}$ generates G. Therefore J cannot be infinite since otherwise we would contradict Corollary 13.62. Therefore every basis in G is finite.

To prove the other part assume that $g_1, g_2, \ldots g_n \in G$ is a basis of G. Then it is easy to verify that $2g_1, 2g_2, \ldots 2g_n \in G$ is a basis of $2G$. Thus we have

$$G = \mathbf{Z}g_1 \oplus \mathbf{Z}g_2 \oplus \ldots \oplus \mathbf{Z}g_n$$
$$2G = \mathbf{Z}2g_1 \oplus \mathbf{Z}2g_2 \oplus \ldots \oplus \mathbf{Z}2g_n$$

and by Theorem 13.49 $G/2G \cong \mathbf{Z}_2^n$. It follows that $|G/2G| = 2^n$, i.e. $n = \log_2(|G/2G|)$. Therefore the number of basis elements in a free group is independent of the choice of a particular basis. □

Example 13.56 will now be approached in a different way. Let $S = \{s_1, s_2, \ldots, s_n\}$ be any finite set of objects. What the objects are does not matter. For example, S may be a class of mathematics students, or as is more relevant to this course, a set of edges or vertices in a graph. The goal is to give meaning to the sum

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n \; , \tag{13.24}$$

where $a_1, a_2, \ldots, a_n$ are integers. For this purpose, let us go back to the definition of the product in (13.5). The product G^n of n copies of G may be identified with the set of all functions f from the finite set $\{1, 2, \ldots, n\}$ to G. Thus an element $(x, y, z) \in G^3$ may be considered as a function $f : \{1, 2, 3\} \to G$ given by $f(1) = x, f(2) = y, f(3) = z$. The group structure is given by pointwise addition: $(f + g)(i) := f(i) + g(i)$. This may be generalized in the following way. First let $\mathbf{Z}^S$ denote the set of all functions $f : S \to \mathbf{Z}$.

Definition 13.65 The *free abelian group generated by a finite set*

$$S = \{s_1, s_2, \ldots s_n\}$$

is the set of all functions $f : S \to \mathbf{Z}$, with the pointwise addition

$$(f + g)(s_i) := f(s_i) + g(s_i), i = 1, 2, ...n \; .$$

Why is this a free abelian group? Consider the functions $\hat{s}_i : S \to \mathbf{Z}, \; i = 1, 2, \ldots, n$ defined by

$$\hat{s}_i(s_j) := \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

Since for any $f \in \mathbf{Z}^S$ we have

$$f = \sum_{i=1}^{n} f(s_i)\widehat{s_i},$$

it is easily verified that $\hat{S} := \{\hat{s_1}, \hat{s_2}, \ldots, \hat{s_n}\}$ is a basis for $\mathbf{Z}^S$. It is called *the canonical basis* and it may be identified with S. Note that if $S = \{1, 2, ...n\}$ we recover $\mathbf{Z}^S = \mathbf{Z}^n$ with the canonical basis $\mathbf{e}_i$.

Example 13.66 Let G be a free abelian group generated by $\{s_1, s_2, \ldots, s_n\}$. Then $G \cong \mathbf{Z}^n$. Indeed, the map $f : \mathbf{Z}^n \to G$ defined on the elements of the canonical basis by $f(\mathbf{e}_i) := s_i$ and extended by linearity is a well defined isomorphism.

Observe that in the case when S is infinite the set $\mathbf{Z}^S$ with the pointwise addition is also a group but it is not generated by $\widehat{S} := \{\widehat{s} \mid s \in S\}$. This is because any linear combination of functions in $\widehat{S}$ may have only a finite number of values different from zero. In particular any non-zero constant function is not generated by $\widehat{S}$. Thus $\mathbf{Z}^S$ cannot be considered as the free abelian group generated by S. Instead we give the following definition.

Definition 13.67 The *free abelian group generated by a possibly infinite set* S is the subgroup of $\mathbf{Z}^S$, consisting of all functions $f : S \to \mathbf{Z}$ satisfying

$$f(s) = 0 \text{ for all but finitely many } s \in S.$$

This group will be denoted by $\mathbf{Z}(S)$. Note that in the case when S is finite the two definitions coincide and we may use both notations.

One can easily verify that, as in the preceding case, any element $f \in \mathbf{Z}(S)$ can be uniquely written as the sum

$$f = \sum_{s \in S} f(s)\widehat{s}.$$

Note that this sum is finite, because $f(s) \neq 0$ for only a finite number of elements $s \in S$.

13.4.2 Subgroups of Free Groups

Although not every abelian group is free, the following important lemma shows that it is easy to produce many free groups from one free group.

Lemma 13.68 *Let F be a finitely generated free abelian group. Then any subgroup H of F is free and the rank of H is less than or equal to the rank of F.*

Proof. Without loss of generality we may assume that $F = \mathbf{Z}^n$ (see Example 13.66). We shall construct a set $\{h_1, h_2, \dots, h_n\}$ of generators of H. We will then extract a linearly independent subset which forms a basis for H. From this it will follow that H is a free group and that the rank of H is less than or equal to the rank of F.

Let $\pi_i : \mathbf{Z}^n \to \mathbf{Z}$ be the canonical projection homomorphism onto the i-th coordinate and let

$$H_m = \{h \in H : \pi_i(h) = 0 \text{ if } i > m\} .$$

Then H_m is a subgroup of H for all $m = 1, 2, \dots, n$ and $H_n = H$.

If $\pi_m(H_m) = 0$, we put $h_m := 0$. If not, then by Lemma 13.13 $\pi_m(H_m)$ is cyclic, so there exists $h_m \in H_m$ such that $\pi_m(h_m)$ generates $\pi_m(H_m)$. It is now proved by induction on m that $\{h_1, h_2, \dots, h_m\}$ generates H_m so, in particular $\{h_1, h_2, \dots, h_n\}$ generates $H_n = H$.

We now want to use a proof by induction on m to show that $\{h_1, \dots, h_m\}$ generates H_m. For $m = 1$, by definition h_1 generates H_1. So assume that $\{h_1, h_2, \dots, h_{m-1}\}$ generates H_{m-1} and let $h \in H_m$. Then $\pi_m(h) = a\pi_m(h_m)$ for some integer a. As a consequence $\pi_m(h - ah_m) = 0$, hence $h - ah_m \in H_{m-1}$. Thus

$$h = ah_m + a_1h_1 + a_2h_2 + \cdots + a_{m-1}h_{m-1}$$

and the conclusion follows.

We will again use an induction proof on m to show that the non zero elements of $\{h_1, h_2 \dots, h_m\}$ constitute a basis of H_m. In this case, the induction assumption can be stated as follows; for fixed m,

$$\alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_mh_m = 0 \;\Rightarrow\; \forall i \in \{1, 2, \ldots m\}\; \alpha_i = 0 \text{ or } h_i = 0. \quad (13.25)$$

For $m = 1$, if $\alpha_1h_1 = 0$ then $\alpha_1 = 0$ or $h_1 = 0$, because $\alpha_1h_1 \in \mathbf{Z}^n$, which is free. Now assume (13.25) is satisfied for some $m < n$. If

$$0 = \alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_{m+1}h_{m+1},$$

then

$$\begin{aligned} 0 &= \pi_{m+1}(\alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_{m+1}h_{m+1}) \quad (13.26) \\ &= \pi_{m+1}(\alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_mh_m) + \alpha_{m+1}\pi_{m+1}(h_{m+1}) \\ &= \alpha_{m+1}\pi_{m+1}(h_{m+1}), \end{aligned}$$

where the last equality follows from the fact that $\alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_mh_m \in H_m$ and $\pi_{m+1}(H_m) = 0$. Therefore either $\alpha_{m+1} = 0$ or $\pi_{m+1}(h_{m+1}) = 0$. In the latter case $\pi_{m+1}(h_{m+1})$ cannot generate $\pi_{m+1}(H_{m+1})$ and by definition $h_{m+1} = 0$. Returning to (13.26) we can now conclude that

$$0 = \alpha_1h_1 + \alpha_2h_2 + \ldots + \alpha_mh_m.$$

By the induction hypothesis, $\alpha_i = 0$ or $h_i = 0$ for each $i = 1, \ldots, m$ and therefore (13.25) holds for $m + 1$. Therefore the non-zero elements of $\{h_1, h_2, \ldots, h_m\}$ form a basis of H. $\square$

Notice that this proof is not constructive, because of the infinite subgroups H_m. Therefore there is no obvious way to turn the above proof into an algorithm. In general constructing a basis of a subgroup of $\mathbf{Z}^n$ may be difficult.

13.4.3 Homomorphisms of Free Groups

Since each free abelian group has a basis, homomorphisms of such groups may be treated very similarly to linear maps of vector spaces. In particular they can be represented as matrices. Therefore, the theory of homomorphisms of free groups is similar to that of linear maps between vector spaces, with one subtle, but significant difference, only matrices with integer coefficients are permitted. We will call such matrices *integer matrices*. We will denote the set of integer matrices with m rows and n columns by $\mathrm{M}_{m,n}(\mathbf{Z})$.

What is the difference between integer matrices and matrices with real coefficients studied in any linear algebra course? There is no problem when we add or multiply integer matrices, because in such a situation the result is always an integer matrix. The same happens when we multiply an integer matrix by any integer.

The problem arises when we want to take the inverse of an integer matrix, as the following example shows.

Example 13.69 Consider the integer matrices

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

One can easily verify that

$$A_1^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

is an integer matrix but

$$A_2^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is not.

We will call an integer matrix $\mathbf{Z}$-*invertible* if its inverse matrix is an integer matrix.

We now turn to homomorphisms of free abelian groups, where integer matrices arise as a natural tool. Let G and H be two finitely generated free abelian groups with bases, respectively, $V := \{v_1, v_2, \ldots, v_n\}$ and $W := \{w_1, w_2, \ldots, w_m\}$. Let $f : G \to H$ be a homomorphism of groups and let $g = \sum_{j=1}^{n} x_j v_j$ be an element of G. Then as in the case of linear maps

$$f(g) = f\left(\sum_{j=1}^{n} x_j v_j\right) = \sum_{j=1}^{n} x_j f(v_j). \tag{13.27}$$

This means that f is uniquely determined by its values on the elements of a basis in G. In particular, in order to define a homomorphism of free groups it is enough to define its values on elements of a selected basis of G.

Example 13.70 Let $f : \mathbf{Z} \to \mathbf{Z}$ be a homomorphism of groups. Since $\mathbf{Z}$ is a free group and $\{1\}$ is a basis for $\mathbf{Z}$, f is determined by its values on 1. If $f(1) = k$ then $f(n) = nk$ for all n. If $k = 0$, f is trivial and $\ker f = \mathbf{Z}$. Otherwise $\ker f = 0$ and $\operatorname{im} f = k\mathbf{Z}$. Since $k\mathbf{Z} = \mathbf{Z}$ if and only if $k = \pm 1$, the only automorphisms of $\mathbf{Z}$ are $\operatorname{id}_{\mathbf{Z}}$ and $-\operatorname{id}_{\mathbf{Z}}$.

On the other hand, the values $f(v_j)$ may be written as linear combinations with integer coefficients of the elements of the basis in H

$$f(v_j) = \sum_{i=1}^{m} b_{ij} w_i \ . \tag{13.28}$$

Therefore, from (13.27) and (13.28) we obtain

$$f(g) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_{ij} x_j\right) w_i \ . \tag{13.29}$$

The integer matrix $B = [b_{ij}]$ is referred to as the *matrix of f in the bases V* and W. Equation (13.29) is equivalent to saying that if $\mathbf{x} = (x_1, x_2, \ldots x_n)$ is the vector of coordinates of g in the basis V and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ is the vector of coordinates of $f(g)$ in the basis W, then $\mathbf{y} = B\mathbf{x}$. Conversely, if $B = [b_{ij}]$ is an $m \times n$ matrix with integer coefficients, then the formula (13.29) defines a unique homomorphism $f : G \to H$.

As in the case of vector spaces, in an arbitrary free group there is usually no natural way to select a basis. Therefore, when speaking about a matrix of a homomorphism, it is important to remember the bases in which the matrix is given.

However, similarly to the case of F^n, one can easily verify that $E^n := \{\mathbf{e}_i^n\}$ is a basis of the group $\mathbf{Z}^n = \prod_{i=1}^{n} \mathbf{Z}$. It is called the *canonical* basis of $\mathbf{Z}^n$. As in the case of vector spaces, a special feature of the canonical basis in $\mathbf{Z}^n$ is the fact that the coordinates of an element $\mathbf{x} = (x_1, x_2, \ldots x_n) \in \mathbf{Z}^n$ in the canonical basis E^n are indistinguishable from the element itself.

As in the case of vector spaces, if $f : \mathbf{Z}^n \to \mathbf{Z}^m$ is a homomorphism, then its matrix in the canonical bases is simply called the matrix of f. We denote this matrix by A_f.

As in the case of linear maps, one easily verifies that if $f' : \mathbf{Z}^m \to \mathbf{Z}^k$ is another homomorphism then

$$A_{f'f} = A_{f'} A_f \tag{13.30}$$

and if $f : \mathbf{Z}^n \to \mathbf{Z}^n$ is an isomorphism of groups, then A_f^{-1} is an integer matrix and

$$A_f^{-1} = A_{f^{-1}}.$$

Exercises

13.15 Let F and G be abelian groups and $f : G \to F$ an epimorphism.

(a) If F is free, show that there exists a subgroup G' of G such that $G = \ker f \oplus G'$. Conclude that $G' \simeq F$.
(b) Give an example showing that if F is not free then the conclusion may be wrong.

13.16 Let $g : H \to G$ be a monomorphism, $f : G \to F$ an epimorphism and suppose that $\ker f = \operatorname{im} g$. If F is free, show that $G \simeq H \oplus F$.

13.17 Let G be a free abelian group with a basis $B = \{g_1, g_2, \ldots g_k, g_{k+1}, \ldots g_n\}$ and let H be its subgroup generated by a subset $B' = \{g_1, g_2, \ldots g_k\}$ of B. Show that G/H is free and the equivalence classes of elements of B' form its basis.

13.18 Let G be a free abelian group, H its subgroup and suppose that G/H is free. Show that $G \simeq H \oplus G/H$.
Hint: Use Exercise 13.16 with the choice of $F := G/H$, $g : H \to G$ the inclusion map and $f : G \to G/H$ given by $f(u) := [u]$.

14

Algorithm Presentation

In this chapter we will describe the language we use in this book to present algorithms. The language we have chosen does not represent any particular programming language, because we do not want to go to far into the technicalities involved in preparing a concrete computer program. The langauge we use is a combination of features found in present day programming languages which we feel are most useful to keep the presentation reasonably detailed but simple. A person with some experience in programming should have no problem in rewriting our algorithms in a concrete programming language.

14.1 Overview

Algorithms accept some data (input), manipulate the data and provide some results (output). The input, intermediate data and output are stored in variables. A *variable* is a container, which can contain some data. Every variable has a name by which the algorithm can assign data to the variable and later read it for further processing. Every algorithm has a name as well, so that it may be used in other algorithms. In this book we assume that a name of a variable or an algorithm may be any sequence of letters and digits starting with a letter.

To access the data stored in variables effectively, the data is organized into some *data structures*. Various data structures are divided into several *data types*. Data types are designed to conveniently store the data needed in particular algorithms. The building of data types begins with some elementary data types upon which more sophisticated data types are constructed.

Often not all data on which an algorithm operates comes from the input. Some data is encoded directly into the algorithm as so called *literals*. These are usually used to introduce the initial values for certain variables. We will discuss data structures and literals used in this book in Section 14.2.

We can restrict a variable to a concrete data type by declaring that variable to be of that data type. This means that an error will occur when data of some other data type will be assigned to that variable. For instance the declaration

int x;

forces an error every time non-integer data is assigned to x. Note the use of boldface. Data type declarations as well as some other language constructs require certain reserved words, called *keywords*. To indicate them clearly we will write them in boldface.

Note that we do not require declaration of variables unless it is not clear from the context what their proper data type should be. This helps to keep the algorithms reasonably short. However one should be aware that in concrete programming languages requiring declaration of variables simplifies memory management and facilitates early detection of possible errors.

Every algorithm presented in this book has the form

function *name*(*list of arguments*)
body

Other algorithms may call a particular algorithm by using its name and supplying the required input data. The comma separated list of arguments assigns names to the data required by the algorithm. On this list we declare the data types to which the arguments must belong.

The body of the algorithm consists of semicolon separated statements. The statements may be simple or compound. A simple statement is the evaluation of an expression. The expression may contain variables, literals, operators, parentheses and calls to other algorithms. Operators may be binary (for instance addition operator + or inequality operator <) and unary (for instance negation operator **not**). The outcome of the expression is usually assigned to a variable by means of the assignment operator as in the example

v := **not** (x + 1 < z);

In the evaluation of expressions we assume the order which follows the standard mathematical conventions. In particular the expressions in parentheses are evaluated first.

As a general rule we assume that the output of the algorithm is the last expression evaluated by the algorithm. However, to emphasize what the algorithm returns we usually use the return statement constructed with the keyword **return** .

return *expression*;

Its only but important effect is that the following expression becomes the last evaluated expression even if some other expressions follow. Therefore the real use of the return statement is in the middle of some branching, when under some conditions we do not want to continue the evaluation of other statements.

The return statement is the simplest example of a compound statement. Other compound statements facilitate branching and loops. They will be discussed in Section 14.3.

We finish this overview with an example of a very simple algorithm

function `average`(**int** `m`, `n`)
return (`m` + `n`)/2;

This algorithm accepts two variables of type integer on input and returns their arithmetic average. Note that we need the literal 2 in this algorithm.

14.2 Data Structures

14.2.1 Elementary Data Types

The basic elementary data type we need is an integer. An explicitly written integer like 2003 or −7 is a literal.

A few variables of the same data type may be declared simultaneously. For instance

int `a`, `v`, `zx`;

declares three integer variables named `a`, `v`, `zx`.

A variable may be declared and initialized at the same time. For instance

int `a` := 121, `v` := 44, `zx`;

declares `a` and assigns to it the initial value 121. Also `v` is declared and initialized to 44 but `zx` is only declared and left uninitialized.

We assume the standard notation for the arithmetic operations on integers, i.e. +,-,*,/. Since the outcome of division of two integers may not be an integer, we usually apply to the result of division the keyword **floor**, which replaces the result by the largest integer not exceeding the exact result. Another operator which we will use with integers is **abs**, which computes the absolute value of the argument.

Sometimes we will work with finite sets whose elements are not integers. In such cases it is convenient to name all elements of such sets and introduce a new elementary data type collecting all the named elements. For example

typedef `endpoint` := (`left`, `right`)

defines a new data type `endpoint`. Now a variable declared by

`endpoint ept`;

may take only values `left` and `right`.

Even more important example of this kind is the following data type

typedef `bool` := (`false`, `true`)

This data type will be used in conditional statements to determine if a branching should occur. We assume that the result of comparison of integers with $<, >, <=, >=, =$ and $\neq$ is of this type. Moreover, we will use the keyword **divides** to determine if an integer divides another integer and we assume that the result is of type `bool`. The three standard logical operations **and**, **or**, **not** may be used with variables of type `bool`.

Occasionally another elementary data type will appear in our algorithms: a string, but only as a literal of the form of a sequence of characters enclosed in double quotes, for instance "Failure". Strings will be used to return information on errors.

14.2.2 Lists

Integer variables and literals may be grouped together to form *lists*. A list is a sequence of items separated by commas and enclosed in parentheses. A list item may be an integer literal, an integer variable or another list. The *length* of a list is the number of its elements.

We will declare a list variable using the keyword **list**. As in the case of an integer variable, a list variable may be initialized at the moment it is declared. For instance

list t := (2002, 11, 7);

declares t to be a list and assigns to it the initial value (2002, 11, 7). A list may be empty. Such a list is denoted by (). Note that both (2002, 11, 7) and () are examples of literals of type list. If m and n are integers then $[m : n]$ will be a shorthand for the list consisting of all consecutive integers from m to n. If $m > n$, then this list is empty.

Given a list t, we may use **length**(t) to obtain the number of elements in the list. Another convenient method to process a list is **cutFirst**(t), which returns a two element list whose first element is the first element of t and the second element is a list consisting of all elements of t but the first element. For instance, if t contains (2002, 11, 7), then **cutFirst**(t) returns (2002, (11, 7)). Note that applying **cutFirst** to an empty list results in an error. An example of using **cutFirst** is given in Section 14.3.2.

As we already mentioned, an item of a list may be another list. However, sometimes it is convenient to put a restriction on the type of items, which a list can store. We can do it using the keyword **of**. For instance

list of int t;

declares a variable t which can store any list whose items are only integers.

A convenient approach is to allow lists of variables on the left hand side of a substitution. Thus in particular

(x, y, z) := (3, 5, 11);

is equivalent to a sequence of substitutions

x := 3; y := 5; z := 11;

14.2.3 Arrays

A special case of a list is an array. An *array* is a list whose items are numbered by consecutive integers. An example of a declaration of an array variable is

array[1 : 5] a;

This declaration means that the variable `a` may only store lists consisting of 5 items numbered from 1 to 5. Individual entries in an array may be accessed by means of the bracket notation, For instance `a`[2] denotes the second item in the array `a`. The number on the left of the colon in the declaration of an array is its *lower limit* or the *first index* and the number on the right is its *upper limit* or the *last index*. The items of an array will often be referred to as *entries*.

An array may be initialized at the moment of declaration. For instance we may write

array$[1:5]$ `a` $:= (15, 7, (2002, 11, 7), 3, -1)$;

This in particular means that the third item of this array is a list, i.e. `a`[3] equals $(2002, 11, 7)$, whereas `a`[5] is an integer -1.

Sometimes it will be convenient to have array elements numbered by integers starting from values other than 1. For instance

array$[-2:1]$ `b` $:= (13, 27, -3, 7)$;

declares an array which can store lists consisting of 4 items numbered $-2, -1, 0, 1$ and initializes `b`$[-2]$ to 13, `b`$[-1]$ to 27, `b`$[0]$ to -3 and `b`$[1]$ to 73.

In some cases it will be useful not to restrict the number of items an array can store. However, to indicate the way the list elements are numbered, we need to give at least one array limit in its declaration. For instance

array$[0:]$ `a`;

declares an array which may store lists of any length and the items of this list are numbered starting from 0.

Every non-empty list may be treated as an array **array**$[1:]$. Thus declaring

list `t`;

we may write `t`[1], `t`[2], `t`[3], etc for the consecutive elements of the list. However using it for a non-existing element is an error.

Given an array `a` we can ask what the limits of that array are by using **firstIndex**(`a`) and **lastIndex**(`a`).

We may use the keyword **of** with arrays. Declaring

array$[1:5]$ **of int** `a`;

we say that `a` is an array which may store five integers numbered from 1 to 5. An item of such an array may only be an integer.

Here is an example of a slightly more sophisticated usage of the keyword **of** .

array$[1:2]$ **of array**$[1:3]$ **of int** `a`;

This declares an array `a` of two entries which themselves are arrays of three integers.

Below we show how such an array may be initialized.

array$[1:2]$ **of array**$[1:3]$ **of int** `a` $:= ((1, 0, 1), (-1, 2, 0))$;

The notation `a`[2][1] referees now to the second list and the first item in this list, which is -1.

Since arrays of arrays will be used frequently, it is convenient to have the following shorthand

array[1 : k][1 : l] a;

meaning

array[1 : k] **of** **array**[1 : l] a;

Of course we can iterate the procedure of adding more and more levels of arrays when necessary.

14.2.4 Vectors and Matrices

The examples we have given so far show that the description of our data structures may become lengthy. For this reason we will use the keyword **typedef** to introduce shorthands. For instance, the data structure which can be used to store integer vectors in $\mathbf{Z}^k$ is

array[1 : k] **of** **int** ;

With

typedef vector[k] := **array**[1 : k] **of** **int** ;

we introduce a shorthand so that

vector[k] v;

is equivalent to writing

array[1 : k] **of** **int** v;

We will represent matrices as arrays of vectors. Thus to store an integer matrix

$$A := \begin{bmatrix} 2 & 7 & 5 \\ 3 & 0 & 1 \end{bmatrix}$$

of 2 rows and 3 columns we will declare the variable A as

array[1 : 3][1 : 2] **of** **int** A := ((2, 3), (7, 0), (5, 1));

In particular A[2][1] will denote the entry in the second row and the first column of A. Notice that in this notation the column index cones first and the row index comes last, which is in the opposite direction to the standard notation concerning matrices but gives us the convenience of easy access to the columns of A. In the case when we want to stick to the traditional order of matrix indices we will write A[i, j] meaning A[j][i].

Similarly to vectors it will be convenient to have the following shorthand for integer matrices

typedef matrix[m, n] := **array**[1 : n][1 : m] **of** **int** ;

Sometimes it will be necessary to extract a submatrix of a matrix. If A is a variable declared as a matrix then A[k : k′, l : l′] denotes the submatrix of A consisting of entries in the i-th row and j-th column such that $\mathtt{k} \leq i \leq \mathtt{k}'$ and $\mathtt{l} \leq j \leq \mathtt{l}'$.

In certain situations it will be convenient not to restrict the length of a vector or the number of rows and columns of a matrix. Then we will use the data types

typedef vector := **array**[1 :] **of int** ;
typedef matrix := **array**[1 :][1 :] **of int** ;

Given a matrix `A` we can access the number of rows and columns of `A` by **numberOfRows**(`A`) and **numberOfColumns**(`A`) respectively.

We also assume that two standard algorithms concerning matrices are at our disposal

function `identityMatrix`(**int** `n`)
function `transpose`(**matrix** `A`)

where the first one returns the $\mathtt{n} \times \mathtt{n}$ identity matrix and the other returns the transpose of the matrix given on input. (see Exercise 14.2).

We allow for the standard + notation for the addition of vectors and matrices and the multiplication matrix by matrix and matrix by vector will be denoted by *.

14.2.5 Sets

The data type *set* is a list which guarantees that no element appears on the list more than once. We will use the keyword **set** to declare the variables of type set. As with ordinary lists, we can use **of** to restrict what type of items the set can contain. Thus

set of int `A` := $(7, 11, 13, 17)$;

declares the variable `A` to be a set of numbers and initializes it with a four element set.

If `A` and `B` are two variables of type **set** then we can use the keywords **union**, **intersection**, **setminus** to obtain the set theoretic union, intersection and difference of the sets stored in these variables. In particular if `B` contains the set $(13, 17, 19)$ then the outcome of **union**(`A`, `B`) is the set $(7, 11, 13, 17, 19)$. To check if A is a subset of B, we will use the construct **subset**(A, B).

To produce the union of a set and an individual item, we will use the keyword **join**. Thus **join**(`A`, 23) will produce the set $(7, 11, 13, 17, 23)$. To test if a given item is an element of a given set we will use the keyword **in** . For example, 13 **in** `A` will result in `true`, whereas 5 **in** `A` will result in `false`.

The keyword **subsets** will be used to produce the set of all subsets of a given set. Thus **subsets**$((1, 2, 3))$ will give

$$((), (1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)).$$

Finally, given two algorithms `f` and `g` and a variable `s` of type set, we assume that the expression

{`f`(`x`) | `x` **in** `s` **and** g(`x`)} evaluates to the set of all results returned by f for arguments x in the set s for which the algorithm g returns true. For instance if s equals $(-2, -3, 0, 3, -6, 7, -11)$, then

$\{\mathbf{abs}(\mathtt{x}) \mid \mathtt{x}\ \mathbf{in}\ \mathtt{s}\ \mathbf{and}\ \mathtt{x} > -5\}$

evaluates to $(2, 3, 0, 7)$. Notice that if the algorithms are very short, then we just use their body instead of introducing and using their names in such constructs.

14.2.6 Hashes

A *hash element* is a list of two items. The first item of a hash element is called the *key* and the other is called the *value*. A *hash* is a list of hash elements such that any two keys are different. We will declare a variable of type hash using the keyword **hash** . For instance

hash h := ((left, 5), (right, 6));

declares a hash variable and initializes it. Hash may be thought of as an array indexed by arbitrary data, not necessarily consecutive integers. Given a key of a hash, the corresponding value may be accessed by means of the braces notation. For instance in the above example h{left} is 5 and writing

h{right} := 8;

will change the value corresponding to the key right from 6 to 8.

Sometimes it will be useful to restrict the possible keys in the hash as well as possible values. For instance using the data type endpoint defined earlier and declaring

hash{endpoint} **of int** I := (3, 4);

provides us with a variable I which can be used to store intervals. The left endpoint of the interval will be stored as the hash value I{left} and the right endpoint as I{right}.

Hashes are often used to save computers memory: if a big array of integers has most items equal to zero, it may be more convenient to store pairs consisting of the position of a non-zero entry and its value. For example, instead of using an array

array[1 : 10] **of int** a := (0, 0, 2, 0, 8, 0, 0, 0, 1, 0);

we can use a hash

hash{ **int** } **of int** b := ((3, 2), (5, 8), (9, 1));

In general hashes are useful in storing associations. We will use them most frequently to store finite functions. The keys in the hash will represent the arguments of the function and the corresponding values in the hash will be the values of the function for the respective arguments. Hashes are much more convenient than arrays for such purposes.

Given a hash h we can use **keys**(h) to obtain a list of all keys in this hash. For instance, if

hash h := ((3, 5), (7, 6), (2, 8));

then **keys**(h) will return $(3, 7, 2)$. Note that by the very definition of a hash, **keys**(h) always evaluates to a set.

In order to remove entirely a key with its value from a hash we will use the keyword **remove**. Thus applying **remove**$(7, \mathtt{h})$ to the above hash will result in the hash containing only $((3, 5), (2, 8))$.

Sometimes it is convenient to be able to check if an item is a key in a hash. For this end we will use the keyword `defined`. Thus **defined**$(3, \mathtt{h})$ will give `true`, whereas **defined**$(6, \mathtt{h})$ will give `false`.

14.3 Compound Statements

14.3.1 Conditional Statements

The conditional statement is used to perform an instruction only if a certain condition is met. It has the form

```
if condition then
    statements;
endif;
```

or the form

```
if condition then
    statements;
else
    statements;
endif;
```

For instance

```
if a divides b then
    x := b/a;
else
    x := 0;
endif;
```

14.3.2 Loop Statements

Loop statements permit repeating some statements certain number of times. We use a few types of loops. The simplest type of a loop is the following

```
for i := m to n do
    statements;
endfor;
```

where the value of the *loop variable* i is increased by 1 from m to n and for every value of i the statements inside the loop are performed. For instance

for $\mathtt{i} := 1$ **to** $\mathtt{n}$ **do**
    $\mathtt{a}[\mathtt{i}] := \mathtt{i} * \mathtt{i}$;
endfor;

This kind of loop may be also used in the following variant

for $\mathtt{i} := \mathtt{n}$ **downto** $\mathtt{m}$ **do**
    statements;
endfor;

in which the loop variable is decreased by 1 instead of being increased.

A special type of loop is used with lists. It has the form

for each x **in** *a list* **do**
    statements;
endfor;

The statements inside this loop are repeated for all the values of the loop variable x taken from the list.

Another type of a loop statement is

while *condition* **do**
    statements;
endwhile ;

where the statements inside the loop are repeated as long as the condition is satisfied. If the condition is not satisfied on entering the loop, no statement inside the loop is performed. For instance

list of int t, s;
...
$\mathtt{s} := ()$;
while $\mathtt{t} \neq ()$ **do**
    $(\mathtt{x}, \mathtt{t}) :=$ **cutFirst**(t);
    if $\mathtt{x} \neq 0$ **then**
        $\mathtt{s} :=$ **join**(x, s);
    endif;
endwhile ;
$\mathtt{t} := \mathtt{s}$;

removes all zero elements from the list t;

A similar loop statement is

repeat
    statements;
until *condition*;

The difference is that the statements inside the loop are performed at least once and the loop is left only after the condition is met.

14.3.3 Keywords break and next

Sometimes it is necessary to proceed with the next pass of the loop, without completing the rest of the statements in the current pass of the loop. We use the **next** keyword for this purpose. When the processing of the loop is to be abandoned entirely, we use the **break** keyword.

In the cases when a few loops are embedded one into another, it may be necessary to use *labels*, to indicate which loop should be broken. For example

```
success := true;
BIGLOOP :
for i := 1 to 5 do
    for j := 1 to 5 do
        if not (q divides a[i, j]) then
            success := false;
            break BIGLOOP;
        endif;
    endfor;
endfor;
```

The same process may be used with the **next** keyword.

14.4 Function and Operator Overloading

Consider the following two algorithms

```
function min( int m, n)
if m < n then
    return m;
endif;
return n;

function min(set of int s)
(m, s) := cutFirst(s);
for each n in (s)
    if n < m then
        m := n;
    endif;
endfor;
return m;
```

The first algorithm computes a minimum of two integers. The other computes the minimum of a set of integers. Notice that both algorithms have the same name. This may be considered an error but actually there is no problem in telling which algorithm we want to use, because the algorithms accept different number of arguments and additionally the arguments must be of different types. Therefore there is no need to use different names. This mechanism of identifying algorithms not only by their name but also by the number and type of arguments is known as *function overloading* and we use it in this book.

When a new data type representing some abstraction of numbers is introduced, one usually writes algorithms allowing for certain arithmetic operations on variables of this new data type. For instance, if `v`, `w` are two variables of type **vector** , then the algorithm, which computes the sum of the two vectors is for instance

```
function add(vector v, w)
m := length(v);
if m ≠ length(v) then
    return "Failure";
endif;
vector u;
for i := 1 to m do
    u[i] := v[i] + w[i];
endfor;
return u;
```

The only problem is that in order to use this algorithm we have to write `add(v, w)` but we would prefer to write simply `v + w`. This may be overcome by introducing the convention that using an operator as "+" in the expression `a + b` is just treated as a shorthand for the call to the algorithm named `operator+` and accepting two arguments of the same type as the variables `a` and `b`. Therefore, to be able to add vectors we only have to rename the above algorithm as `operator+`. This mechanism of adding operators for new data structures is known as *operator overloading* and we also use it in this book.

In particular, this allows us to use the standard + notation for the addition of vectors and matrices and the standard "*" notation for the multiplication of a matrix by a matrix and a matrix by a vector. (See Exercise 14.3).

14.5 Analysis of Algorithms

Algorithms are mathematical objects as any other mathematical stuff. In particular one proves theorems about algorithms. The first thing one wants to prove about any algorithm is that the algorithms does not loop infinitely many times but always returns some result or at least returns a result for input data satisfying some conditions. Sometimes this fact is obvious, sometimes the proof is lengthy and complicated. Actually, the only reason the algorithm may loop infinitely is because of the presence of certain loops. For instance, if the algorithms contains

```
int i := 1;
while i > 1 do
    i := i + 1;
endwhile ;
```

then obviously it will loop infinitely many times. However, when the loops are nested and conditions under which the loops are completed are complicated, proving that the algorithm actually always returns may be a difficult task.

The next thing is to show that the result is not an error. We presented several situations under which we assumed that the algorithm terminates with an error, for instance we may try to assign a list to a variable of type integer. This is usually done together with proving what are the features of the object the algorithm returns. Algorithms are built for a certain purpose and we want to know that they actually do what they were designed for. This must be proved too. Again, often this is quite obvious but often is not.

As an example consider the algorithm **function** `min`(**set of int** `s`) presented in Section 14.4. First notice, that there is only one loop in this algorithm and it always completes. Therefore the algorithm always completes. We expect that the algorithm returns the minimal element of the set of integers on input and this is almost true. Actually this is true under the condition that the set on the input is non-empty. Indeed, if the set is empty, then the algorithm terminates with error, because in this case **cutFirst** fails by definition. Thus consider the case when `s` contains a non-empty set $(x_0, x_1, \ldots x_p)$. Let m_0 denote the value of the `m` variable after the first assignment and for $i = 1, 2, \ldots p$ let m_i denote the value of the `m` variable after the i-th pass of the **for** loop. We will show by induction that

$$m_i = \min\{x_0, x_1, \ldots x_i\} \tag{14.1}$$

for all $i = 0, 1, 2, \ldots p$. For $i = 0$ the right hand of (14.1) reduces to $\{x_0\}$ and the left hand is just x_0, so the first induction step is obvious. Thus assume (14.1) is satisfied for some $i < p$. There are two cases possible. Either $x_{i+1} < m_i$ and then $m_{i+1} = x_{i+1}$ or $x_{i+1} \geq m_i$ and then $m_{i+1} = m_i$. In both cases it is obvious from the induction assumption that the equation (14.1) remains true for i replaced by $i + 1$. In particular, after the last path of the **for** loop `m` contains the minimum of the whole set. Thus we proved that under the assumption that the argument `s` contains a nonempty set, the algorithm returns its minimal element.

Once it is proved that the algorithm returns what is claimed, the algorithm may be used in the mathematically rigorous sense, i.e. applying the algorithm to some concrete input we may proclaim the features of the output as a theorem.

There is one more thing people prove about algorithms: statements about their complexity, i.e how does the time needed for the algorithm to complete depend on the size of the input. In general this is a difficult question. For many algorithms we can only tell the worst case estimates but from the experience we know that in many cases the algorithms perform much better. In this book we are mainly concerned in showing that some problems of infinite mathematics may be reduced to problems which may be answered rigorously by algorithmic means. Moreover we know that these algorithms do solve concrete problems for which no other solution is known in reasonable time. The question how to solve these problems most efficiently is left for another book.

Exercises

14.1 Write an algorithm which computes the maximum of two integers and an algorithm which computes the maximum of a set of integers.

14.2 Write the bodies of the algorithms `identityMatrix` and `transpose` presented at the end of Section 14.2.4.

14.3 Write algorithms overloading the operators "+", "-" and "*" so that they can be used with variables of type **vector** and **matrix** in the standard mathematical sense.

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